

Convex Optimization

Euclidean Distance Geometry^{2ε}

In my career, I found that the best people are the ones that really understand the content, and they're a pain in the butt to manage. But you put up with it because they're so great at the content. And that's what makes great products; it's not process, it's content.

—Steve Jobs, 1995

1	Overview	21
2	Convex geometry	33
2.1	Convex set	33
2.2	Vectorized-matrix inner product	45
2.3	Hulls	54
2.4	Halfspace, Hyperplane	63
2.5	Subspace representations	75
2.6	Extreme, Exposed	81
2.7	Cones	85
2.8	Cone boundary	93
2.9	Positive semidefinite (PSD) cone	99
2.10	Conic independence (c.i.)	121
2.11	When extreme means exposed	126
2.12	Convex polyhedra	127
2.13	Dual cone & generalized inequality	132
3	Geometry of convex functions	185
3.1	Convex function	185
3.2	Practical norm functions, absolute value	190
3.3	Powers, roots, and inverted functions	198
3.4	Affine function	200
3.5	Epigraph, Sublevel set	204
3.6	Gradient	210
3.7	Convex matrix-valued function	221
3.8	Quasiconvex	226
3.9	Salient properties	237

4	Semidefinite programming	239
4.1	Conic problem	240
4.2	Framework	247
4.3	Rank reduction	259
4.4	Rank-constrained semidefinite program	265
4.5	Constraining cardinality	291
4.6	Cardinality and rank constraint examples	308
4.7	Constraining rank of indefinite matrices	346
4.8	Convex Iteration rank-1	351
5	Euclidean Distance Matrix	357
5.1	EDM	358
5.2	First metric properties	359
5.3	$\exists$ fifth Euclidean metric property	359
5.4	EDM definition	363
5.5	Invariance	390
5.6	Injectivity of $\mathbf{D}$ & unique reconstruction	393
5.7	Embedding in affine hull	398
5.8	Euclidean metric <i>versus</i> matrix criteria	402
5.9	Bridge: Convex polyhedra to EDMs	408
5.10	EDM-entry composition	414
5.11	EDM indefiniteness	417
5.12	List reconstruction	423
5.13	Reconstruction examples	425
5.14	Fifth property of Euclidean metric	431
6	Cone of distance matrices	439
6.1	Defining EDM cone	440
6.2	Polyhedral bounds	443
6.3	$\sqrt{\text{EDM}}$ cone is not convex	443
6.4	EDM definition in $\mathbf{11}^T$	444
6.5	Correspondence to PSD cone $\mathbb{S}_+^{N-1}$	451
6.6	Vectorization & projection interpretation	456
6.7	A geometry of completion	460
6.8	Dual EDM cone	467
6.9	Theorem of the alternative	480
6.10	Postscript	480
7	Proximity problems	481
7.1	First prevalent problem:	487
7.2	Second prevalent problem:	495
7.3	Third prevalent problem:	504
7.4	Conclusion	512

A Linear algebra	513
A.1 Main-diagonal δ operator, λ , tr , vec	513
A.2 Semidefiniteness: domain of test	517
A.3 Proper statements of positive semidefiniteness	519
A.4 Schur complement	528
A.5 Eigenvalue decomposition	533
A.6 Singular value decomposition, SVD	536
A.7 Zeros	541
B Simple matrices	547
B.1 Rank-one matrix (dyad)	547
B.2 Doublet	552
B.3 Elementary matrix	552
B.4 Auxiliary V -matrices	554
B.5 Orthomatrices	558
C Some analytical optimal results	563
C.1 Properties of infima	563
C.2 Trace, singular and eigen values	564
C.3 Orthogonal Procrustes problem	570
C.4 Two-sided orthogonal Procrustes	572
C.5 Nonconvex quadratics	576
D Matrix calculus	577
D.1 Directional derivative, Taylor series	577
D.2 Tables of gradients and derivatives	594
E Projection	603
E.1 Idempotent matrices	606
E.2 $I - P$, Projection on algebraic complement	610
E.3 Symmetric idempotent matrices	611
E.4 Algebra of projection on affine subsets	616
E.5 Projection examples	617
E.6 Vectorization interpretation,	624
E.7 Projection on matrix subspaces	630
E.8 Range/Rowspace interpretation	633
E.9 Projection on convex set	633
E.10 Alternating projection	645
F Notation, Definitions, Glossary	663
Bibliography	679
Index	703

List of Figures

1	Overview	21
1	Sigma delta quantizer	22
2	Room geometry estimation by first acoustic reflections	22
3	<i>Orion nebula</i>	23
4	Application of trilateration is localization	24
5	Molecular conformation	25
6	Face recognition	26
7	<i>Swiss roll</i>	27
8	USA map reconstruction	28
9	Honeycomb, Hexabenzocoronene molecule	29
10	Robotic vehicles	30
11	Reconstruction of David	32
12	David by distance geometry	32
2	Convex geometry	33
13	Slab	35
14	Open, closed, convex sets	37
15	Intersection of line with boundary	39
16	Tangentials	41
17	Inverse image	44
18	Inverse image under linear map	45
19	<i>Tesseract</i>	47
20	Linear injective mapping of Euclidean body	49
21	Linear noninjective mapping of Euclidean body	50
22	Convex hull of a random list of points	54
23	Hulls	56
24	Two Fantopes	58
25	Nuclear Norm Ball	60
26	Convex hull of rank-1 matrices	61
27	A simplicial cone	64
28	Hyperplane illustrated $\partial\mathcal{H}$ is a partially bounding line	65
29	Hyperplanes in $\mathbb{R}^2$	67

30	Affine independence	70
31	$\{z \in \mathcal{C} \mid a^T z = \kappa_i\}$	71
32	Hyperplane supporting closed set	72
33	Minimizing hyperplane over affine set in nonnegative orthant	79
34	Maximizing hyperplane over convex set	80
35	Closed convex set illustrating exposed and extreme points	83
36	Two-dimensional nonconvex cone	85
37	Nonconvex cone made from lines	86
38	Nonconvex cone is convex cone boundary	86
39	Union of convex cones is nonconvex cone	86
40	Truncated nonconvex cone $\mathcal{X}$	87
41	Cone exterior is convex cone	87
42	Not a cone	89
43	Minimum element, Minimal element	91
44	$\mathcal{K}$ is a pointed polyhedral cone not full-dimensional	94
45	Exposed and extreme directions	98
46	Positive semidefinite cone	100
47	Convex Schur-form set	102
48	Projection of truncated PSD cone	104
49	Circular cone showing axis of revolution	113
50	Circular section	114
51	Polyhedral inscription	115
52	Conically (in)dependent vectors	122
53	Pointed six-faceted polyhedral cone and its dual	124
54	Minimal set of generators for halfspace about origin	125
55	Venn diagram for cones and polyhedra	129
56	Range form polyhedron	129
57	Simplex	131
58	Two views of a simplicial cone and its dual	133
59	Two equivalent constructions of dual cone	136
60	Dual polyhedral cone construction by right angle	137
61	$\mathcal{K}$ is a halfspace about the origin	138
62	Iconic primal and dual objective functions	139
63	Orthogonal cones	143
64	Blades $\mathcal{K}$ and $\mathcal{K}^*$	145
65	Membership w.r.t $\mathcal{K}$ and orthant	153
66	Shrouded polyhedral cone	156
67	Simplicial cone $\mathcal{K}$ in $\mathbb{R}^2$ and its dual	160
68	Monotone nonnegative cone $\mathcal{K}_{\mathcal{M}+}$ and its dual	169
69	Monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual	170
70	Two views of monotone cone $\mathcal{K}_{\mathcal{M}}$ and its dual	171
71	First-order optimality condition	174

3	Geometry of convex functions	185
72	Convex functions having unique minimizer	187
73	Minimum/Minimal element, dual cone characterization	188
74	1-norm ball $\mathcal{B}_1$ from compressed sensing/compressive sampling	192
75	Cardinality minimization, signed <i>versus</i> unsigned variable	193
76	1-norm variants	194
77	Affine function	202
78	Supremum of affine functions	203
79	Epigraph	204
80	Quadratic bowl gradient in $\mathbb{R}^2$ evaluated on grid	211
81	Quadratic function convexity in terms of its gradient	217
82	Contour plot of convex real function at selected levels	219
83	Iconic quasiconvex function	226
84	Quasiconcave monotonic function xu	228
85	Arbitrary magnitude analog filter design	234
4	Semidefinite programming	239
86	Venn diagram of convex program classes	242
87	Visualizing positive semidefinite cone in high dimension	244
88	Primal/Dual transformations	252
89	Projection <i>versus</i> convex iteration	268
90	Trace heuristic	268
91	Sensor-network localization	271
92	2-lattice of sensors and anchors for localization example	274
93	3-lattice of sensors and anchors for localization example	275
94	4-lattice of sensors and anchors for localization example	276
95	5-lattice of sensors and anchors for localization example	277
96	Uncertainty ellipsoids orientation and eccentricity	278
97	2-lattice localization solution	281
98	3-lattice localization solution	281
99	4-lattice localization solution	282
100	5-lattice localization solution	282
101	10-lattice localization solution	283
102	100 randomized noiseless sensor localization	283
103	100 randomized sensors localization	284
104	Nonnegative spectral factorization	287
105	Regularization curve for convex iteration	290
106	1-norm heuristic	293
107	Sparse sampling theorem	296
108	Signal dropout	299
109	Signal dropout reconstruction	300
110	Simplex with intersecting line problem in compressed sensing	302
111	Geometric interpretations of sparse-sampling constraints	305
112	Permutation matrix column-norm and column-sum constraint	312
113	MAX CUT problem	318

114	Shepp-Logan phantom	323
115	MRI radial sampling pattern in Fourier domain	327
116	Aliased phantom	328
117	Neighboring-pixel stencil on Cartesian grid	330
118	Differentiable almost everywhere	331
119	<i>Eternity</i> II	333
120	<i>Eternity</i> II game-board grid	335
121	<i>Eternity</i> II demo-game piece illustrating edge-color ordering	336
122	<i>Eternity</i> II vectorized demo-game-board piece descriptions	337
123	<i>Eternity</i> II difference and boundary parameter construction	339
124	Sparsity pattern for composite <i>Eternity</i> II variable matrix	340
125	MIT logo	348
126	One-pixel camera	349
127	One-pixel camera - compression estimates	349
128	Straight line through three direction vectors by midpoint fit	355
5	Euclidean Distance Matrix	357
129	Convex hull of three points	358
130	Complete dimensionless <i>EDM graph</i>	360
131	Fifth Euclidean metric property	362
132	<i>Fermat point</i>	369
133	Arbitrary hexagon in $\mathbb{R}^3$	371
134	Kissing number	372
135	<i>Trilateration</i>	376
136	This EDM graph provides unique isometric reconstruction	380
137	Two sensors $\bullet$ and three anchors $\circ$	380
138	Two discrete linear trajectories of sensors	381
139	Coverage in cellular telephone network	383
140	Contours of equal signal power	383
141	Depiction of molecular conformation	385
142	Square diamond	392
143	Orthogonal complements in $\mathbb{S}^N$ abstractly oriented	394
144	Elliptope $\mathcal{E}^3$	410
145	Elliptope $\mathcal{E}^2$ interior to $\mathbb{S}_+^2$	411
146	Smallest eigenvalue of $-V_N^T D V_N$	415
147	Some entrywise EDM compositions	415
148	Map of United States of America	426
149	Largest ten eigenvalues of $-V_N^T O V_N$	428
150	<i>Relative-angle inequality tetrahedron</i>	433
151	Nonsimplicial pyramid in $\mathbb{R}^3$	437

6 Cone of distance matrices	439
152 Relative boundary of cone of Euclidean distance matrices	441
153 Example of $V_{\mathcal{X}}$ selection to make an EDM	446
154 Vector $V_{\mathcal{X}}$ spirals	448
155 Three views of translated negated ellipsope	454
156 Halfline $\mathcal{T}$ on PSD cone boundary	457
157 Vectorization and projection interpretation example	458
158 Intersection of EDM cone with hyperplane	461
159 Neighborhood graph	462
160 <i>Trefoil knot</i> untied	464
161 <i>Trefoil ribbon</i>	466
162 Orthogonal complement of geometric center subspace	470
163 EDM cone construction by flipping PSD cone	471
164 Decomposing member of polar EDM cone	474
165 Ordinary dual EDM cone projected on $\mathbb{S}_h^3$	479
 7 Proximity problems	 481
166 Pseudo-Venn diagram for EDM	483
167 Elbow placed in path of projection	484
168 Convex envelope	499
 A Linear algebra	 513
169 Geometrical interpretation of full SVD	539
 B Simple matrices	 547
170 Four fundamental subspaces for any dyad	548
171 Four fundamental subspaces for doublet	552
172 Four fundamental subspaces for elementary matrix	553
173 Gimbal	560
 D Matrix calculus	 577
174 Convex quadratic bowl in $\mathbb{R}^2 \times \mathbb{R}$	586
 E Projection	 603
175 Action of pseudoinverse	604
176 Nonorthogonal projection of $x \in \mathbb{R}^3$ on $\mathcal{R}(U) = \mathbb{R}^2$	609
177 Biorthogonal expansion of point $x \in \text{aff } \mathcal{K}$	619
178 Linear regression <i>versus</i> principal component analysis	623
179 Dual interpretation of projection on convex set	635
180 Projection on orthogonal complement	638
181 Projection on dual cone	640
182 Projection product on convex set in subspace	644

183	von Neumann-style projection of point b	647
184	Alternating projection on two halfspaces	648
185	Distance, feasibility, optimization	649
186	Alternating projection on nonnegative orthant and hyperplane	652
187	Geometric convergence of iterates in norm	652
188	Distance between PSD cone and iterate in $\mathcal{A}$	656
189	Dykstra's alternating projection algorithm	657
190	Polyhedral normal cones	658
191	Normal cone to elliptope	660
192	Normal-cone progression	661

List of Tables

2 Convex geometry

Table [2.9.2.3.1](#), rank *versus* dimension of $\mathbb{S}_+^3$ faces [107](#)

Table [2.10.0.0.1](#), maximum number of c.i. directions [122](#)

Cone Table **1** [166](#)

Cone Table **S** [167](#)

Cone Table **A** [168](#)

Cone Table **1*** [172](#)

4 Semidefinite programming

Faces of $\mathbb{S}_+^3$ corresponding to faces of $\mathcal{S}_+^3$ [245](#)

5 Euclidean Distance Matrix

Précis [5.7.2](#): affine dimension in terms of rank [401](#)

B Simple matrices

Auxiliary V -matrix Table [B.4.4](#) [558](#)

D Matrix calculus

Table [D.2.1](#), algebraic gradients and derivatives [595](#)

Table [D.2.2](#), trace Kronecker gradients [596](#)

Table [D.2.3](#), trace gradients and derivatives [597](#)

Table [D.2.4](#), logarithmic determinant gradients, derivatives [599](#)

Table [D.2.5](#), determinant gradients and derivatives [600](#)

Table [D.2.6](#), logarithmic derivatives [600](#)

Table [D.2.7](#), exponential gradients and derivatives [601](#)