# Convex Optimization
# Euclidean Distance Geometry $^{2\mathcal{E}}$

*In my career, I found that the best people are the ones that really understand the content, and they're a pain in the butt to manage. But you put up with it because they're so great at the content. And that's what makes great products; it's not process, it's content.*

–Steve Jobs, 1995


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