Chapter 6

Cone of distance matrices

For $N>3$, the cone of EDMs is no longer a circular cone and the geometry becomes complicated...

−Hayden, Wells, Liu, & Tarazaga, 1991 [201, §3]

In the subspace of symmetric matrices $\mathbb{S}^N$, we know that the convex cone of Euclidean distance matrices $\mathbb{EDM}^N$ (the EDM cone) does not intersect the positive semidefinite cone $\mathbb{S}^N_+$ (PSD cone) except at the origin, their only vertex; there can be no positive or negative semidefinite EDM. (1187) [251]

$$\mathbb{EDM}^N \cap \mathbb{S}^N_+ = \mathbf{0} \tag{1264}$$

Even so, the two convex cones can be related. In §6.8.1 we prove the equality

$$\mathbb{EDM}^N = \mathbb{S}^N_h \cap \left(\mathbb{S}^{N\perp}_c - \mathbb{S}^N_+\right) \tag{1357}$$

a resemblance to EDM definition (976) where

$$\mathbb{S}^N_h = \left\{A \in \mathbb{S}^N \mid \delta(A) = \mathbf{0}\right\} \tag{66}$$

is the symmetric hollow subspace (§2.2.3) and where

$$\mathbb{S}^{N\perp}_c = \left\{u\mathbf{1}^{\mathrm{T}} + \mathbf{1}u^{\mathrm{T}} \mid u \in \mathbb{R}^N\right\} \tag{2115}$$

is the orthogonal complement of the geometric center subspace (§E.7.2.0.2)

$$\mathbb{S}^N_c = \left\{Y \in \mathbb{S}^N \mid Y\mathbf{1} = \mathbf{0}\right\} \tag{2113}$$

6.0.1 gravity

Equality (1357) is equally important as the known isomorphisms (1095) (1096) (1107) (1108) relating the EDM cone $\mathbb{EDM}^N$ to positive semidefinite cone $\mathbb{S}_+^{N-1}$ (§5.6.2.1) or to an $N(N-1)/2$-dimensional face of $\mathbb{S}_+^N$ (§5.6.1.1).[6.1] But those isomorphisms have never led to this equality relating whole cones $\mathbb{EDM}^N$ and $\mathbb{S}_+^N$.

Equality (1357) is not obvious from the various EDM definitions such as (976) or (1280) because inclusion must be proved algebraically in order to establish equality; $\mathbb{EDM}^N \supseteq \mathbb{S}_h^N \cap \left(\mathbb{S}_c^{N\perp} - \mathbb{S}_+^N\right)$. We will instead prove (1357) using purely geometric methods.

6.0.2 highlight

In §6.8.1.7 we show: the Schoenberg criterion for discriminating Euclidean distance matrices

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; \left\{\begin{array}{c} -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \in \mathbb{S}_+^{N-1} \\ D \in \mathbb{S}_h^N \end{array}\right. \tag{995}$$

is a discretized membership relation (§2.13.4, dual generalized inequalities) between the EDM cone and its ordinary dual.

6.1 Defining EDM cone

We invoke a popular matrix criterion to illustrate correspondence between the EDM and PSD cones belonging to the ambient space of symmetric matrices:

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; \left\{\begin{array}{c} -V D V \in \mathbb{S}_+^N \\ D \in \mathbb{S}_h^N \end{array}\right. \tag{999}$$

where $V \in \mathbb{S}^N$ is the geometric centering matrix (§B.4). The set of all EDMs of dimension $N \times N$ forms a closed convex cone $\mathbb{EDM}^N$ because any pair of EDMs satisfies the definition of a convex cone (175); *videlicet*, for each and every $\zeta_1, \zeta_2 \geq 0$ (§A.3.1.0.2)

$$\begin{array}{c} \zeta_1 V D_1 V + \zeta_2 V D_2 V \succeq 0 \\ \zeta_1 D_1 + \zeta_2 D_2 \in \mathbb{S}_h^N \end{array} \;\Leftarrow\; \begin{array}{cc} V D_1 V \succeq 0\,, & V D_2 V \succeq 0 \\ D_1 \in \mathbb{S}_h^N\,, & D_2 \in \mathbb{S}_h^N \end{array} \tag{1265}$$

and convex cones are invariant to inverse linear transformation [325, p.22].

[6.1] Because both positive semidefinite cones are frequently in play, dimension is explicit.

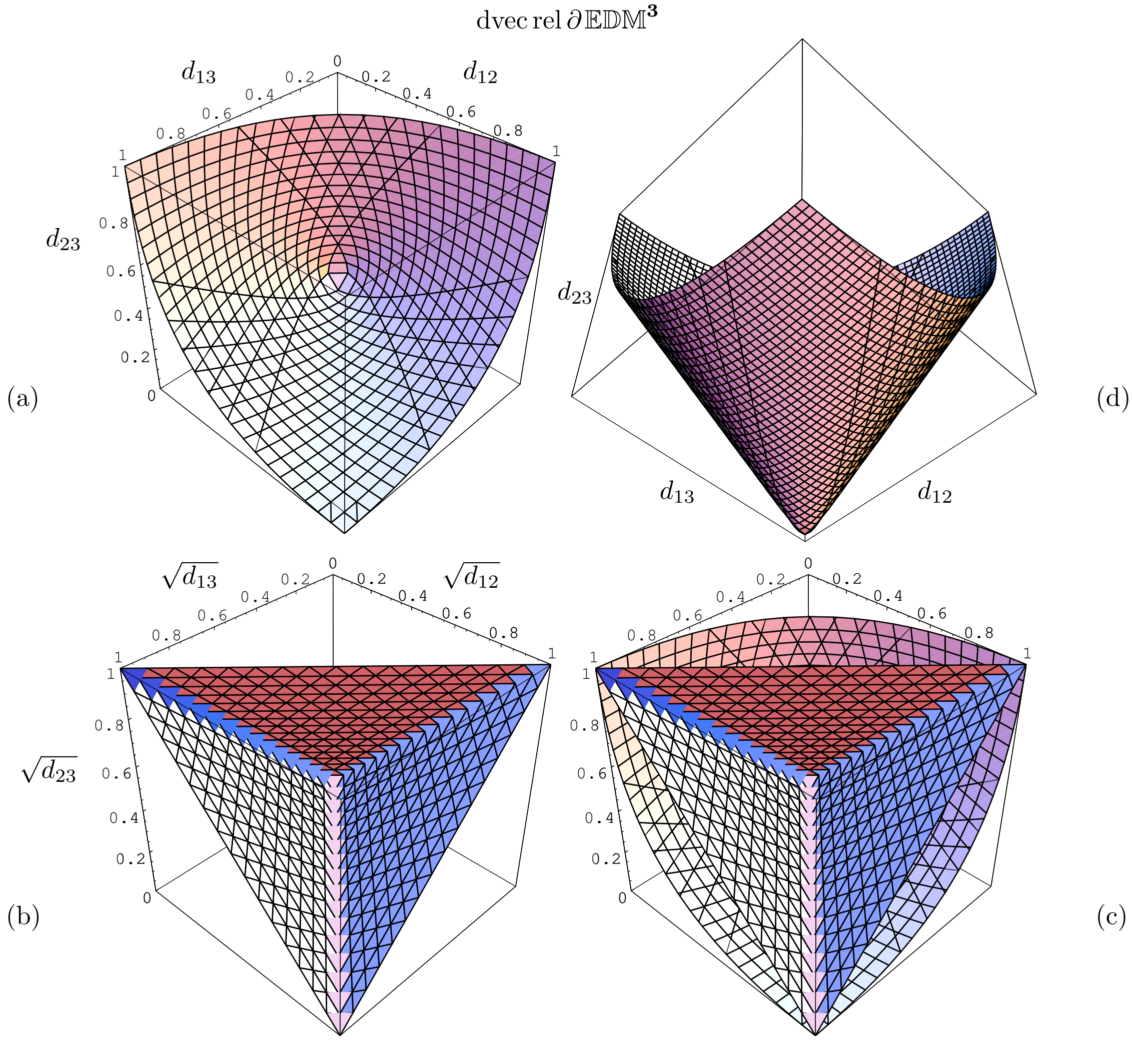


Figure 152: Relative boundary (tiled) of EDM cone $\mathbb{EDM}^3$ drawn truncated in isometrically isomorphic subspace $\mathbb{R}^3$. **(a)** EDM cone drawn in usual distance-square coordinates d_{ij}. View is from interior toward origin. Unlike positive semidefinite cone, EDM cone is not selfdual; neither is it proper in ambient symmetric subspace (dual EDM cone for this example belongs to isomorphic $\mathbb{R}^6$). **(b)** Drawn in its natural coordinates $\sqrt{d_{ij}}$ (absolute distance), cone remains convex (*confer* §5.10); intersection of three halfspaces (1150) whose partial boundaries each contain origin. Cone geometry becomes nonconvex (nonpolyhedral) in higher dimension. (§6.3) **(c)** Two coordinate systems artificially superimposed. Coordinate transformation from d_{ij} to $\sqrt{d_{ij}}$ appears a topological contraction. **(d)** Sitting on its vertex $\mathbf{0}$, pointed $\mathbb{EDM}^3$ is a circular cone having axis of revolution $\operatorname{dvec}(-E) = \operatorname{dvec}(\mathbf{1}\mathbf{1}^{\mathrm{T}} - I)$ (1183) (73). (Rounded vertex is plot artifact.)

6.1.0.0.1 Definition. *Cone of Euclidean distance matrices.*
In the subspace of symmetric matrices, the set of all Euclidean distance matrices forms a unique immutable pointed closed convex cone called the *EDM cone*: for $N>0$

$$\begin{aligned}\mathbb{EDM}^N &\triangleq \left\{D\in\mathbb{S}_h^N \mid -VDV\in\mathbb{S}_+^N\right\}\\ &= \bigcap_{z\in\mathcal{N}(\mathbf{1}^{\mathrm{T}})}\left\{D\in\mathbb{S}^N \mid \langle zz^{\mathrm{T}},-D\rangle\geq 0\,,\quad \delta(D)=\mathbf{0}\right\}\end{aligned}\tag{1266}$$

The EDM cone in isomorphic $\mathbb{R}^{N(N+1)/2}$ [*sic*] is the intersection of an infinite number (when $N>2$) of halfspaces about the origin and a finite number of hyperplanes through the origin in vectorized variable $D=[d_{ij}]$. Hence $\mathbb{EDM}^N$ is not full-dimensional with respect to $\mathbb{S}^N$ because it is confined to the symmetric hollow subspace $\mathbb{S}_h^N$. The EDM cone relative interior comprises

$$\begin{aligned}\operatorname{rel\,int}\mathbb{EDM}^N &= \bigcap_{z\in\mathcal{N}(\mathbf{1}^{\mathrm{T}})}\left\{D\in\mathbb{S}^N \mid \langle zz^{\mathrm{T}},-D\rangle> 0\,,\quad \delta(D)=\mathbf{0}\right\}\\ &= \left\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=N-1\right\}\end{aligned}\tag{1267}$$

while its relative boundary comprises

$$\begin{aligned}\operatorname{rel}\partial\mathbb{EDM}^N &= \left\{D\in\mathbb{EDM}^N \mid \langle zz^{\mathrm{T}},-D\rangle=0 \text{ for some } z\in\mathcal{N}(\mathbf{1}^{\mathrm{T}})\right\}\\ &= \left\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)<N-1\right\}\end{aligned}\tag{1268}$$

△

This cone is more easily visualized in the isomorphic vector subspace $\mathbb{R}^{N(N-1)/2}$ corresponding to $\mathbb{S}_h^N$:

In the case $N=1$ point, the EDM cone is the origin in $\mathbb{R}^0$.

In the case $N=2$, the EDM cone is the nonnegative real line in $\mathbb{R}$; a halfline in a subspace of the realization in Figure **156**.

The EDM cone in the case $N=3$ is a circular cone in $\mathbb{R}^{\mathbf{3}}$ illustrated in Figure **152**(a)(d); rather, the set of all matrices

$$D=\begin{bmatrix}0 & d_{12} & d_{13}\\ d_{12} & 0 & d_{23}\\ d_{13} & d_{23} & 0\end{bmatrix}\in\mathbb{EDM}^{\mathbf{3}}\tag{1269}$$

makes a circular cone in this dimension. In this case, the first four Euclidean metric properties are necessary and sufficient tests to certify realizability of triangles; (1245). Thus triangle inequality property 4 describes three halfspaces (1150) whose intersection makes a polyhedral cone in $\mathbb{R}^{\mathbf{3}}$ of realizable $\sqrt{d_{ij}}$ (absolute distance); an isomorphic subspace representation of the set of all EDMs D in the natural coordinates

$$\sqrt[\circ]{D}\triangleq\begin{bmatrix}0 & \sqrt{d_{12}} & \sqrt{d_{13}}\\ \sqrt{d_{12}} & 0 & \sqrt{d_{23}}\\ \sqrt{d_{13}} & \sqrt{d_{23}} & 0\end{bmatrix}\tag{1270}$$

illustrated in Figure **152**b.

6.2 Polyhedral bounds

The convex cone of EDMs is nonpolyhedral in d_{ij} for $N>2$; *e.g*, Figure **152**a. Still we found necessary and sufficient bounding polyhedral relations consistent with EDM cones for cardinality $N=1,2,3,4$:

$N=3$. Transforming distance-square coordinates d_{ij} by taking their positive square root provides the polyhedral cone in Figure **152**b; polyhedral because an intersection of three halfspaces in natural coordinates $\sqrt{d_{ij}}$ is provided by triangle inequalities (1150). This polyhedral cone implicitly encompasses necessary and sufficient metric properties: nonnegativity, selfdistance, symmetry, and triangle inequality.

$N=4$. Relative-angle inequality (1251) together with four Euclidean metric properties are necessary and sufficient tests for realizability of tetrahedra. (1252) Albeit relative angles θ_{ikj} (1042) are nonlinear functions of the d_{ij}, relative-angle inequality provides a regular tetrahedron in $\mathbb{R}^{\mathbf{3}}$ [*sic*] (Figure **150**) bounding angles θ_{ikj} at vertex x_k consistently with $\mathbb{EDM}^{\mathbf{4}}$.[6.2]

Yet were we to employ the procedure outlined in §5.14.3 for making generalized triangle inequalities, then we would find all the necessary and sufficient d_{ij}-transformations for generating bounding polyhedra consistent with EDMs of any higher dimension ($N>3$).

6.3 $\sqrt{\text{EDM}}$ cone is not convex

For some applications, like a molecular conformation problem (Figure **5**, Figure **141**) or multidimensional scaling [109] [373], absolute distance $\sqrt{d_{ij}}$ is the preferred variable. Taking square root of the entries in all EDMs D of dimension N, we get another cone but not a convex cone when $N>3$ (Figure **152**b): [93, §4.5.2]

$$\sqrt{\mathbb{EDM}^N} \triangleq \{\sqrt[\circ]{D} \mid D\in\mathbb{EDM}^N\} \tag{1271}$$

where $\sqrt[\circ]{D}$ is defined like (1270). It is a cone simply because any cone is completely constituted by rays emanating from the origin: (§2.7) Any given ray $\{\zeta\Gamma\in\mathbb{R}^{N(N-1)/2} \mid \zeta\geq 0\}$ remains a ray under entrywise square root: $\{\sqrt[\circ]{\zeta\Gamma}\in\mathbb{R}^{N(N-1)/2} \mid \zeta\geq 0\}$. It is already established that

$$D\in\mathbb{EDM}^N \;\Rightarrow\; \sqrt[\circ]{D}\in\mathbb{EDM}^N \qquad (1182)$$

But because of how $\sqrt{\mathbb{EDM}^N}$ is defined, it is obvious that (*confer* §5.10)

$$D\in\mathbb{EDM}^N \;\Leftrightarrow\; \sqrt[\circ]{D}\in\sqrt{\mathbb{EDM}^N} \tag{1272}$$

Were $\sqrt{\mathbb{EDM}^N}$ convex, then given $\sqrt[\circ]{D_1}\,,\sqrt[\circ]{D_2}\in\sqrt{\mathbb{EDM}^N}$ we would expect their conic combination $\sqrt[\circ]{D_1}+\sqrt[\circ]{D_2}$ to be a member of $\sqrt{\mathbb{EDM}^N}$. That is easily proven false by

[6.2] Still, property-4 triangle inequalities (1150) corresponding to each principal 3×3 submatrix of $-V_{\mathcal{N}}^{\mathrm{T}}DV_{\mathcal{N}}$ demand that the corresponding $\sqrt{d_{ij}}$ belong to a polyhedral cone like that in Figure **152**b.

counterexample via (1272), for then $(\sqrt[\circ]{D_1}+\sqrt[\circ]{D_2})\circ(\sqrt[\circ]{D_1}+\sqrt[\circ]{D_2})$ would need to be a member of $\mathbb{EDM}^N$.

Notwithstanding,

$$\sqrt{\mathbb{EDM}^N} \subseteq \mathbb{EDM}^N \tag{1273}$$

by (1182) (Figure **152**), and we learn how to transform a nonconvex *proximity problem* in the natural coordinates $\sqrt{d_{ij}}$ to a convex optimization in §7.2.1.

6.4 EDM definition in $\mathbf{1}\mathbf{1}^{\mathrm{T}}$

Any EDM D corresponding to affine dimension r has representation

$$\mathbf{D}(V_{\mathcal{X}}) \triangleq \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})^{\mathrm{T}} - 2V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}} \in \mathbb{EDM}^N \tag{1274}$$

where $\mathcal{R}(V_{\mathcal{X}}\in\mathbb{R}^{N\times r})\subseteq\mathcal{N}(\mathbf{1}^{\mathrm{T}}) = \mathbf{1}^{\perp}$

$$V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}} = \delta^2(V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}}) \quad \textbf{and} \quad V_{\mathcal{X}} \text{ is full-rank with orthogonal columns.} \tag{1275}$$

Equation (1274) is simply the standard EDM definition (976) with a centered list X as in (1063); Gram matrix $X^{\mathrm{T}}X$ has been replaced with the subcompact singular value decomposition (§A.6.2)[6.3]

$$V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}} \equiv V^{\mathrm{T}}X^{\mathrm{T}}XV \in \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1276}$$

This means: inner product $V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}}$ is an $r\times r$ diagonal matrix Σ of nonzero singular values.

Vector $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})$ may me decomposed into complementary parts by projecting it on orthogonal subspaces $\mathbf{1}^{\perp}$ and $\mathcal{R}(\mathbf{1})$: namely,

$$P_{\mathbf{1}^{\perp}}\big(\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\big) = V\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) \tag{1277}$$

$$P_{\mathbf{1}}\big(\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\big) = \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) \tag{1278}$$

Of course

$$\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) = V\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) + \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) \tag{1279}$$

by (998). Substituting this into EDM definition (1274), we get the Hayden, Wells, Liu, & Tarazaga EDM formula [201, §2]

$$\mathbf{D}(V_{\mathcal{X}}\,,\,y) \triangleq y\mathbf{1}^{\mathrm{T}} + \mathbf{1}y^{\mathrm{T}} + \frac{\lambda}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} - 2V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}} \in \mathbb{EDM}^N \tag{1280}$$

where

$$\lambda \triangleq 2\|V_{\mathcal{X}}\|_{\mathrm{F}}^2 = \mathbf{1}^{\mathrm{T}}\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})2 \quad \textbf{and} \quad y \triangleq \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) - \frac{\lambda}{2N}\mathbf{1} = V\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) \tag{1281}$$

[6.3] Subcompact SVD: $V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}} \triangleq Q\sqrt{\Sigma}\sqrt{\Sigma}Q^{\mathrm{T}} \equiv V^{\mathrm{T}}X^{\mathrm{T}}XV$. So $V_{\mathcal{X}}^{\mathrm{T}}$ is not necessarily XV (§5.5.1.0.1), although affine dimension $r = \operatorname{rank}(V_{\mathcal{X}}^{\mathrm{T}}) = \operatorname{rank}(XV)$. (1120)

and $y=\mathbf{0}$ if and only if $\mathbf{1}$ is an eigenvector of EDM D. Scalar λ becomes an eigenvalue when corresponding eigenvector $\mathbf{1}$ exists.[6.4]

Then the particular dyad sum from (1280)

$$y\mathbf{1}^{\mathrm{T}}+\mathbf{1}y^{\mathrm{T}}+\frac{\lambda}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}\in\mathbb{S}_c^{N\perp} \tag{1282}$$

must belong to the orthogonal complement of the geometric center subspace (p.632), whereas $V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}\in\mathbb{S}_c^N\cap\mathbb{S}_+^N$ (1276) belongs to the positive semidefinite cone in the geometric center subspace.

Proof. We validate eigenvector $\mathbf{1}$ and eigenvalue λ.
($\Rightarrow$) Suppose $\mathbf{1}$ is an eigenvector of EDM D. Then because

$$V_{\mathcal{X}}^{\mathrm{T}}\mathbf{1}=\mathbf{0} \tag{1283}$$

it follows

$$\begin{aligned} D\mathbf{1} \;=\; \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\mathbf{1}^{\mathrm{T}}\mathbf{1}+\mathbf{1}\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})^{\mathrm{T}}\mathbf{1} \;&=\; N\,\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})+\|V_{\mathcal{X}}\|_{\mathrm{F}}^2\mathbf{1} \\ &\Rightarrow\; \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\propto\mathbf{1} \end{aligned} \tag{1284}$$

For some $\kappa\in\mathbb{R}_+$

$$\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})^{\mathrm{T}}\mathbf{1}=N\kappa=\operatorname{tr}(V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}})=\|V_{\mathcal{X}}\|_{\mathrm{F}}^2 \quad\Rightarrow\quad \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})=\frac{1}{N}\|V_{\mathcal{X}}\|_{\mathrm{F}}^2\mathbf{1} \tag{1285}$$

so $y=\mathbf{0}$.
($\Leftarrow$) Now suppose $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})=\dfrac{\lambda}{2N}\mathbf{1}$; *id est*, $y=\mathbf{0}$. Then

$$D=\frac{\lambda}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}-2V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}\in\mathbb{EDM}^N \tag{1286}$$

$\mathbf{1}$ is an eigenvector with corresponding eigenvalue λ. ♦

6.4.1 Range of EDM D

From §B.1.1 pertaining to linear independence of dyad sums: If the transpose halves of all the dyads in the sum (1274)[6.5] make a linearly independent set, then the nontranspose halves constitute a basis for the range of EDM D. Saying this mathematically: For $D\in\mathbb{EDM}^N$

$$\begin{aligned} \mathcal{R}(D)&=\mathcal{R}([\,\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\quad\mathbf{1}\quad V_{\mathcal{X}}\,]) &&\Leftarrow\quad \operatorname{rank}([\,\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\quad\mathbf{1}\quad V_{\mathcal{X}}\,])=2+r \\ \mathcal{R}(D)&=\mathcal{R}([\,\mathbf{1}\quad V_{\mathcal{X}}\,]) &&\Leftarrow\quad \text{otherwise} \end{aligned} \tag{1287}$$

[6.4] *e.g.*, when $X=I$ in EDM definition (976).
[6.5] Identifying columns $V_{\mathcal{X}}\triangleq[\,v_1\cdots v_r\,]$, then $V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}=\sum_i v_iv_i^{\mathrm{T}}$ is also a sum of dyads.

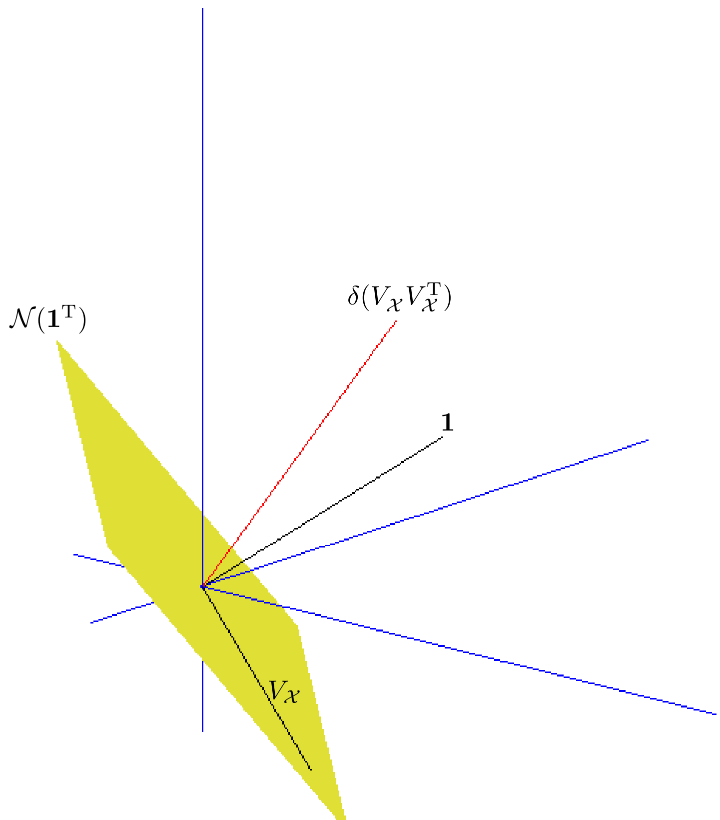


Figure 153: Example of $V_{\mathcal{X}}$ selection to make an EDM corresponding to cardinality $N=3$ and affine dimension $r=1$; $V_{\mathcal{X}}$ is a vector in nullspace $\mathcal{N}(\mathbf{1}^{\mathrm{T}})\subset\mathbb{R}^{\mathbf{3}}$. Nullspace of $\mathbf{1}^{\mathrm{T}}$ is hyperplane in $\mathbb{R}^{\mathbf{3}}$ (drawn truncated) having normal $\mathbf{1}$. Vector $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})$ may or may not be in plane spanned by $\{\mathbf{1}\,,V_{\mathcal{X}}\}$, but belongs to nonnegative orthant which is strictly supported by $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$.

6.4.1.0.1 Proof. We need that condition under which the rank equality is satisfied: We know $\mathcal{R}(V_{\mathcal{X}})\perp\mathbf{1}$, but what is the relative geometric orientation of $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})$? $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\succeq 0$ because $V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}\succeq 0$, and $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\propto\mathbf{1}$ remains possible (1284); this means $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})\notin\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ simply because it has no negative entries. (Figure **153**) If the projection of $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})$ on $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ does not belong to $\mathcal{R}(V_{\mathcal{X}})$, then that is a necessary and sufficient condition for linear independence (l.i.) of $\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})$ with respect to $\mathcal{R}([\mathbf{1}\quad V_{\mathcal{X}}])$; *id est*,

$$\begin{aligned}
V\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) &\neq V_{\mathcal{X}}\,a \quad \text{for any } a\in\mathbb{R}^r\\
(I-\tfrac{1}{N}\mathbf{11}^{\mathrm{T}})\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}) &\neq V_{\mathcal{X}}\,a\\
\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})-\tfrac{1}{N}\|V_{\mathcal{X}}\|_{\mathrm{F}}^2\mathbf{1} &\neq V_{\mathcal{X}}\,a\\
\delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}})-\tfrac{\lambda}{2N}\mathbf{1} = y &\neq V_{\mathcal{X}}\,a \;\Leftrightarrow\; \{\mathbf{1},\;\; \delta(V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}),\;\; V_{\mathcal{X}}\} \text{ is l.i.}
\end{aligned}\tag{1288}$$

When this condition is violated (when (1281) $y=V_{\mathcal{X}}\,a_{\mathrm{p}}$ for some particular $a\in\mathbb{R}^r$), on the other hand, then from (1280) we have

$$\begin{aligned}
\mathcal{R}\big(D=y\mathbf{1}^{\mathrm{T}}+\mathbf{1}y^{\mathrm{T}}+\tfrac{\lambda}{N}\mathbf{11}^{\mathrm{T}}-2V_{\mathcal{X}}V_{\mathcal{X}}^{\mathrm{T}}\big) &= \mathcal{R}\big((V_{\mathcal{X}}\,a_{\mathrm{p}}+\tfrac{\lambda}{N}\mathbf{1})\mathbf{1}^{\mathrm{T}}+(\mathbf{1}a_{\mathrm{p}}^{\mathrm{T}}-2V_{\mathcal{X}})V_{\mathcal{X}}^{\mathrm{T}}\big)\\
&= \mathcal{R}([\,V_{\mathcal{X}}\,a_{\mathrm{p}}+\tfrac{\lambda}{N}\mathbf{1}\quad \mathbf{1}a_{\mathrm{p}}^{\mathrm{T}}-2V_{\mathcal{X}}\,])\\
&= \mathcal{R}([\,\mathbf{1}\quad V_{\mathcal{X}}\,])
\end{aligned}\tag{1289}$$

An example of such a violation is (1286) where, in particular, $a_{\mathrm{p}}=\mathbf{0}$. ♦

Then a statement parallel to (1287) is, for $D\in\mathbb{EDM}^N$ (Theorem 5.7.3.0.1)

$$\begin{aligned}
\operatorname{rank}(D) = r+2 \quad&\Leftrightarrow\quad y\notin\mathcal{R}(V_{\mathcal{X}}) \quad \big(\Leftrightarrow\; \mathbf{1}^{\mathrm{T}}D^{\dagger}\mathbf{1}=0\big)\\
\operatorname{rank}(D) = r+1 \quad&\Leftrightarrow\quad y\in\mathcal{R}(V_{\mathcal{X}}) \quad \big(\Leftrightarrow\; \mathbf{1}^{\mathrm{T}}D^{\dagger}\mathbf{1}\neq 0\big)
\end{aligned}\tag{1290}$$

6.4.2 Boundary constituents of EDM cone

Expression (1274) has utility in forming the set of all EDMs corresponding to affine dimension r:

$$\begin{aligned}
&\Big\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=r\Big\}\\
&=\Big\{\mathbf{D}(V_{\mathcal{X}}) \mid V_{\mathcal{X}}\in\mathbb{R}^{N\times r},\;\operatorname{rank}V_{\mathcal{X}}=r\,,\; V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}}=\delta^2(V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}})\,,\;\mathcal{R}(V_{\mathcal{X}})\subseteq\mathcal{N}(\mathbf{1}^{\mathrm{T}})\Big\}
\end{aligned}\tag{1291}$$

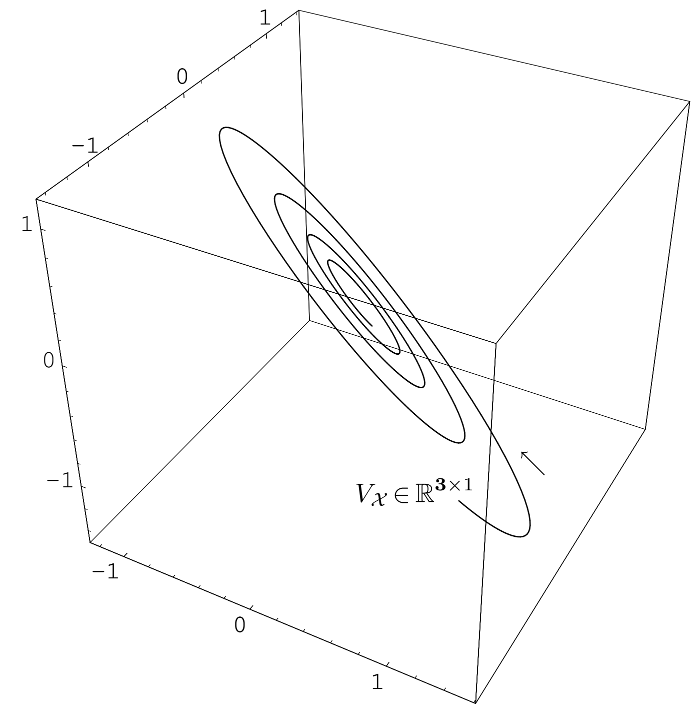
whereas $\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)\leq r\}$ is the closure of this same set;

$$\Big\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)\leq r\Big\}=\overline{\Big\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=r\Big\}}\tag{1292}$$

For example,

$$\begin{aligned}
\operatorname{rel}\partial\mathbb{EDM}^N &= \Big\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)<N-1\Big\}\\
&= \bigcup_{r=0}^{N-2}\Big\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=r\Big\}
\end{aligned}\tag{1293}$$

(a)



(b)

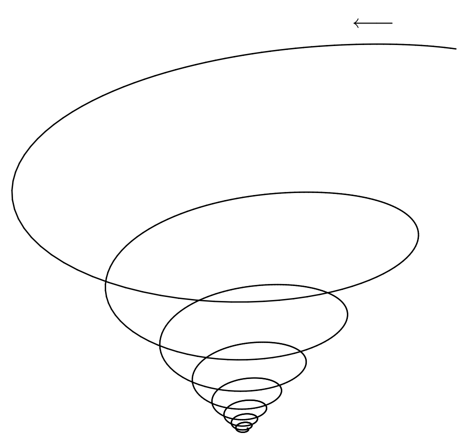


Figure 154: **(a)** Vector $V_{\mathcal{X}}$ from Figure **153** spirals in $\mathcal{N}(\mathbf{1}^{\mathrm{T}}) \subset \mathbb{R}^{\mathbf{3}}$ decaying toward origin. (Spiral is two-dimensional in vector space $\mathbb{R}^{\mathbf{3}}$.) **(b)** Corresponding trajectory $\mathbf{D}(V_{\mathcal{X}})$ on EDM cone relative boundary creates a vortex also decaying toward origin. There are two complete orbits on EDM cone boundary about axis of revolution for every single revolution of $V_{\mathcal{X}}$ about origin. (Vortex is three-dimensional in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$.)

None of these are necessarily convex sets, although

$$\begin{aligned}\mathbb{EDM}^N &= \bigcup_{r=0}^{N-1}\left\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=r\right\}\\ &= \overline{\left\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=N-1\right\}}\\ \operatorname{rel\,int}\mathbb{EDM}^N &= \left\{D\in\mathbb{EDM}^N \mid \operatorname{rank}(VDV)=N-1\right\}\end{aligned}\tag{1294}$$

are pointed convex cones.

When cardinality $N=3$ and affine dimension $r=2$, for example, the relative interior $\operatorname{rel\,int}\mathbb{EDM}^{\mathbf{3}}$ is realized via (1291). (§6.5)

When $N=3$ and $r=1$, the relative boundary of the EDM cone $\operatorname{dvec}\operatorname{rel}\partial\mathbb{EDM}^{\mathbf{3}}$ is realized in isomorphic $\mathbb{R}^{\mathbf{3}}$ as in Figure **152**d. This figure could be constructed via (1292) by spiraling vector $V_{\mathcal{X}}$ tightly about the origin in $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$; as can be imagined with aid of Figure **153**. Vectors close to the origin in $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ are correspondingly close to the origin in $\mathbb{EDM}^N$. As vector $V_{\mathcal{X}}$ orbits the origin in $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$, the corresponding EDM orbits the axis of revolution while remaining on the boundary of the circular cone $\operatorname{dvec}\operatorname{rel}\partial\mathbb{EDM}^{\mathbf{3}}$. (Figure **154**)

6.4.3 Faces of EDM cone

Like the positive semidefinite cone, EDM cone faces are EDM cones.

6.4.3.0.1 Exercise. *Isomorphic faces.*
Prove that in high cardinality N, any set of EDMs made via (1291) or (1292) with particular affine dimension r is isomorphic with any set admitting the same affine dimension but made in lower cardinality. ▼

6.4.3.1 smallest face that contains an EDM

Now suppose we are given a particular EDM $\mathbf{D}(V_{\mathcal{X}_\mathrm{p}})\in\mathbb{EDM}^N$ corresponding to affine dimension r and parametrized by $V_{\mathcal{X}_\mathrm{p}}$ in (1274). The EDM cone's smallest face that contains $\mathbf{D}(V_{\mathcal{X}_\mathrm{p}})$ is

$$\begin{aligned}&\mathcal{F}\Big(\mathbb{EDM}^N \ni \mathbf{D}(V_{\mathcal{X}_\mathrm{p}})\Big)\\ &\quad= \overline{\left\{\mathbf{D}(V_{\mathcal{X}}) \mid V_{\mathcal{X}}\in\mathbb{R}^{N\times r},\ \operatorname{rank}V_{\mathcal{X}}=r\,,\ V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}}=\delta^2(V_{\mathcal{X}}^{\mathrm{T}}V_{\mathcal{X}})\,,\ \mathcal{R}(V_{\mathcal{X}})\subseteq\mathcal{R}(V_{\mathcal{X}_\mathrm{p}})\right\}}\\ &\quad\simeq \mathbb{EDM}^{r+1}\end{aligned}\tag{1295}$$

which is isomorphic[6.6] with convex cone $\mathbb{EDM}^{r+1}$, hence of dimension

$$\dim\mathcal{F}\Big(\mathbb{EDM}^N \ni \mathbf{D}(V_{\mathcal{X}_\mathrm{p}})\Big) = (r+1)r/2\tag{1296}$$

[6.6] The fact that the smallest face is isomorphic with another EDM cone (perhaps smaller than $\mathbb{EDM}^N$) is implicit in [201, §2].

in isomorphic $\mathbb{R}^{N(N-1)/2}$. Not all dimensions are represented; *e.g*, the EDM cone has no two-dimensional faces.

When cardinality $N=4$ and affine dimension $r=2$ so that $\mathcal{R}(V_{\mathcal{X}_\mathrm{p}})$ is any two-dimensional subspace of three-dimensional $\mathcal{N}(\mathbf{1}^\mathrm{T})$ in $\mathbb{R}^4$, for example, then the corresponding face of $\mathbb{EDM}^4$ is isometrically isomorphic with: (1292)

$$\mathbb{EDM}^3 = \{D \in \mathbb{EDM}^3 \mid \operatorname{rank}(VDV) \leq 2\} \simeq \mathcal{F}(\mathbb{EDM}^4 \ni \mathbf{D}(V_{\mathcal{X}_\mathrm{p}})) \tag{1297}$$

Each two-dimensional subspace of $\mathcal{N}(\mathbf{1}^\mathrm{T})$ corresponds to another three-dimensional face.

Because each and every principal submatrix of an EDM in $\mathbb{EDM}^N$ (§5.14.3) is another EDM [251, §4.1], for example, then each principal submatrix belongs to a particular face of $\mathbb{EDM}^N$.

6.4.3.2 extreme directions of EDM cone

In particular, extreme directions (§2.8.1) of $\mathbb{EDM}^N$ correspond to affine dimension $r=1$ and are simply represented: for any particular cardinality $N \geq 2$ (§2.8.2) and each and every nonzero vector z in $\mathcal{N}(\mathbf{1}^\mathrm{T})$

$$\begin{aligned}\Gamma &\triangleq (z \circ z)\mathbf{1}^\mathrm{T} + \mathbf{1}(z \circ z)^\mathrm{T} - 2zz^\mathrm{T} \in \mathbb{EDM}^N \\ &= \delta(zz^\mathrm{T})\mathbf{1}^\mathrm{T} + \mathbf{1}\delta(zz^\mathrm{T})^\mathrm{T} - 2zz^\mathrm{T}\end{aligned} \tag{1298}$$

is an extreme direction corresponding to a one-dimensional face of the EDM cone $\mathbb{EDM}^N$ that is a ray in isomorphic subspace $\mathbb{R}^{N(N-1)/2}$.

> Proving this would exercise the fundamental definition (186) of extreme direction. Here is a sketch: Any EDM may be represented
>
> $$\mathbf{D}(V_\mathcal{X}) = \delta(V_\mathcal{X} V_\mathcal{X}^\mathrm{T})\mathbf{1}^\mathrm{T} + \mathbf{1}\delta(V_\mathcal{X} V_\mathcal{X}^\mathrm{T})^\mathrm{T} - 2V_\mathcal{X} V_\mathcal{X}^\mathrm{T} \in \mathbb{EDM}^N \tag{1274}$$
>
> where matrix $V_\mathcal{X}$ (1275) has orthogonal columns. For the same reason (1556) that zz^T is an extreme direction of the positive semidefinite cone (§2.9.2.7) for any particular nonzero vector z, there is no conic combination of distinct EDMs (each conically independent of Γ (§2.10)) equal to Γ. ■

6.4.3.2.1 Example. *Biorthogonal expansion of an EDM.* (*confer* §2.13.7.1.1)
When matrix D belongs to the EDM cone, nonnegative coordinates for biorthogonal expansion are the eigenvalues $\lambda \in \mathbb{R}^N$ of $-VDV\frac{1}{2}$: For any $D \in \mathbb{S}_h^N$ it holds

$$D = \delta\big(-VDV\tfrac{1}{2}\big)\mathbf{1}^\mathrm{T} + \mathbf{1}\delta\big(-VDV\tfrac{1}{2}\big)^\mathrm{T} - 2\big(-VDV\tfrac{1}{2}\big) \tag{1087}$$

By diagonalization $-VDV\frac{1}{2} \triangleq Q\Lambda Q^\mathrm{T} \in \mathbb{S}_c^N$ (§A.5.1) we may write

$$\begin{aligned}D &= \delta\Big(\sum_{i=1}^N \lambda_i\, q_i q_i^\mathrm{T}\Big)\mathbf{1}^\mathrm{T} + \mathbf{1}\delta\Big(\sum_{i=1}^N \lambda_i\, q_i q_i^\mathrm{T}\Big)^\mathrm{T} - 2\sum_{i=1}^N \lambda_i\, q_i q_i^\mathrm{T} \\ &= \sum_{i=1}^N \lambda_i \left(\delta(q_i q_i^\mathrm{T})\mathbf{1}^\mathrm{T} + \mathbf{1}\delta(q_i q_i^\mathrm{T})^\mathrm{T} - 2q_i q_i^\mathrm{T}\right)\end{aligned} \tag{1299}$$

where q_i is the i^{th} eigenvector of $-VDV\frac{1}{2}$ arranged columnar in orthogonal matrix

$$Q = [\, q_1 \;\; q_2 \;\; \cdots \;\; q_N \,] \in \mathbb{R}^{N\times N} \tag{399}$$

and where $\{\delta(q_i q_i^{\mathrm T})\mathbf{1}^{\mathrm T} + \mathbf{1}\,\delta(q_i q_i^{\mathrm T})^{\mathrm T} - 2q_i q_i^{\mathrm T}\,,\;\; i=1\ldots N\}$ are extreme directions of some pointed polyhedral cone $\mathcal{K}\subset\mathbb{S}_h^N$ and extreme directions of $\mathbb{EDM}^N$. Invertibility of (1299)

$$\begin{aligned}
-VDV\tfrac{1}{2} &= -V\sum_{i=1}^{N}\lambda_i\left(\delta(q_i q_i^{\mathrm T})\mathbf{1}^{\mathrm T} + \mathbf{1}\,\delta(q_i q_i^{\mathrm T})^{\mathrm T} - \,2q_i q_i^{\mathrm T}\right)V\tfrac{1}{2}\\
&= \sum_{i=1}^{N}\lambda_i\, q_i q_i^{\mathrm T}
\end{aligned} \tag{1300}$$

implies linear independence of those extreme directions. Then biorthogonal expansion is expressed

$$\operatorname{dvec} D \;=\; Y Y^{\dagger}\operatorname{dvec} D \;=\; Y\,\lambda\big(-VDV\tfrac{1}{2}\big) \tag{1301}$$

where

$$Y \triangleq \left[\operatorname{dvec}\!\big(\delta(q_i q_i^{\mathrm T})\mathbf{1}^{\mathrm T} + \mathbf{1}\,\delta(q_i q_i^{\mathrm T})^{\mathrm T} - \,2q_i q_i^{\mathrm T}\big)\,,\;\; i=1\ldots N\right] \in \mathbb{R}^{N(N-1)/2\times N} \tag{1302}$$

When D belongs to the EDM cone in the subspace of symmetric hollow matrices, unique coordinates $Y^{\dagger}\operatorname{dvec} D$ for this biorthogonal expansion must be the nonnegative eigenvalues λ of $-VDV\frac{1}{2}$. This means D simultaneously belongs to the EDM cone and to the pointed polyhedral cone $\operatorname{dvec}\mathcal{K} = \operatorname{cone}(Y)$. $\square$

6.4.3.3 open question

Result (1296) is analogous to that for the positive semidefinite cone (222), although the question remains open whether all faces of $\mathbb{EDM}^N$ (whose dimension is less than dimension of the cone) are exposed like they are for the positive semidefinite cone.[6.7] (§2.9.2.3) [365]

6.5 Correspondence to PSD cone $\mathbb{S}_+^{N-1}$

Hayden, Wells, Liu, & Tarazaga [201, §2] assert one-to-one correspondence of EDMs with positive semidefinite matrices in the symmetric subspace. Because $\operatorname{rank}(VDV)\leq N-1$ (§5.7.1.1), that PSD cone corresponding to the EDM cone can only be $\mathbb{S}_+^{N-1}$. [9, §18.2.1] To clearly demonstrate this correspondence, we invoke inner-product form EDM definition

$$\begin{gathered}
\mathbf{D}(\Phi) \;=\; \begin{bmatrix} 0 \\ \delta(\Phi)\end{bmatrix}\mathbf{1}^{\mathrm T} + \mathbf{1}\begin{bmatrix} 0 & \delta(\Phi)^{\mathrm T}\end{bmatrix} \;-\; 2\begin{bmatrix} 0 & \mathbf{0}^{\mathrm T} \\ \mathbf{0} & \Phi\end{bmatrix} \in \mathbb{EDM}^N\\
\Leftrightarrow\\
\Phi \succeq 0
\end{gathered} \tag{1105}$$

Then the EDM cone may be expressed

$$\mathbb{EDM}^N = \left\{\mathbf{D}(\Phi) \mid \Phi\in\mathbb{S}_+^{N-1}\right\} \tag{1303}$$

[6.7] Elementary example of face not exposed is given by the closed convex set in Figure 35 and in Figure 45.

Hayden & Wells' assertion can therefore be equivalently stated in terms of an inner-product form EDM operator

$$\mathbf{D}(\mathbb{S}_+^{N-1}) = \mathbb{EDM}^N \qquad (1107)$$
$$\mathbf{V}_{\mathcal{N}}(\mathbb{EDM}^N) = \mathbb{S}_+^{N-1} \qquad (1108)$$

identity (1108) holding because $\mathcal{R}(V_{\mathcal{N}}) = \mathcal{N}(\mathbf{1}^{\mathrm{T}})$ (983), linear functions $\mathbf{D}(\Phi)$ and $\mathbf{V}_{\mathcal{N}}(D) = -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}}$ (§5.6.2.1) being mutually inverse.

In terms of affine dimension r, Hayden & Wells claim particular correspondence between PSD and EDM cones:

$r = N-1$: Symmetric hollow matrices $-D$ positive definite on $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ correspond to points relatively interior to the EDM cone.

$r < N-1$: Symmetric hollow matrices $-D$ positive semidefinite on $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$, where $-V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}}$ has at least one 0 eigenvalue, correspond to points on the relative boundary of the EDM cone.

$r = 1$: Symmetric hollow nonnegative matrices rank-one on $\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ correspond to extreme directions (1298) of the EDM cone; *id est*, for some nonzero vector u (§A.3.1.0.7)

$$\left.\begin{array}{r}\operatorname{rank} V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} = 1 \\ D \in \mathbb{S}_h^N \cap \mathbb{R}_+^{N\times N}\end{array}\right\} \Leftrightarrow \begin{array}{c} D \in \mathbb{EDM}^N \\ D \text{ is an extreme direction}\end{array} \Leftrightarrow \left\{\begin{array}{c} -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \equiv uu^{\mathrm{T}} \\ D \in \mathbb{S}_h^N \end{array}\right. \qquad (1304)$$

6.5.0.0.1 Proof. Case $r=1$ is easily proved: From the nonnegativity development in §5.8.1, extreme direction (1298), and Schoenberg criterion (995), we need show only sufficiency; *id est*, prove

$$\left.\begin{array}{r}\operatorname{rank} V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} = 1 \\ D \in \mathbb{S}_h^N \cap \mathbb{R}_+^{N\times N}\end{array}\right\} \Rightarrow \begin{array}{c} D \in \mathbb{EDM}^N \\ D \text{ is an extreme direction}\end{array}$$

Any symmetric matrix D satisfying the rank condition must have the form, for $z, q \in \mathbb{R}^N$ and nonzero $z \in \mathcal{N}(\mathbf{1}^{\mathrm{T}})$,

$$D = \pm(\mathbf{1}q^{\mathrm{T}} + q\mathbf{1}^{\mathrm{T}} - 2zz^{\mathrm{T}}) \qquad (1305)$$

because (§5.6.2.1, *confer* §E.7.2.0.2)

$$\mathcal{N}(\mathbf{V}_{\mathcal{N}}(D)) = \{\mathbf{1}q^{\mathrm{T}} + q\mathbf{1}^{\mathrm{T}} \mid q \in \mathbb{R}^N\} \subseteq \mathbb{S}^N \qquad (1306)$$

Hollowness demands $q = \delta(zz^{\mathrm{T}})$ while nonnegativity demands choice of positive sign in (1305). Matrix D thus takes the form of an extreme direction (1298) of the EDM cone. ♦

The foregoing proof is not extensible in rank: An EDM with corresponding affine dimension r has the general form, for $\{z_i \in \mathcal{N}(\mathbf{1}^{\mathrm{T}}),\ i = 1 \ldots r\}$ an independent set,

$$D = \mathbf{1}\,\delta\Big(\sum_{i=1}^r z_i z_i^{\mathrm{T}}\Big)^{\mathrm{T}} + \delta\Big(\sum_{i=1}^r z_i z_i^{\mathrm{T}}\Big)\mathbf{1}^{\mathrm{T}} - 2\sum_{i=1}^r z_i z_i^{\mathrm{T}} \in \mathbb{EDM}^N \qquad (1307)$$

The EDM so defined relies principally on the sum $\sum z_i z_i^{\mathrm T}$ having positive summand coefficients ($\Leftrightarrow -V_{\mathcal N}^{\mathrm T} D V_{\mathcal N} \succeq 0$)[6.8]. Then it is easy to find a sum incorporating negative coefficients while meeting rank, nonnegativity, and symmetric hollowness conditions but not positive semidefiniteness on subspace $\mathcal{R}(V_{\mathcal N})$; *e.g*, from page 418,

$$-V\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix} V\frac{1}{2} \;=\; z_1 z_1^{\mathrm T} - z_2 z_2^{\mathrm T} \tag{1308}$$

6.5.0.0.2 Example. *Extreme rays* versus *rays on the boundary.*

The EDM $D = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ is an extreme direction of $\mathbb{EDM}^{\mathbf 3}$ where $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in (1304). Because $-V_{\mathcal N}^{\mathrm T} D V_{\mathcal N}$ has eigenvalues $\{0, 5\}$, the ray whose direction is D also lies on the relative boundary of $\mathbb{EDM}^{\mathbf 3}$.

In exception, EDM $D = \kappa\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, for any particular $\kappa > 0$, is an extreme direction of $\mathbb{EDM}^{\mathbf 2}$ but $-V_{\mathcal N}^{\mathrm T} D V_{\mathcal N}$ has only one eigenvalue: $\{\kappa\}$. Because $\mathbb{EDM}^{\mathbf 2}$ is a ray whose relative boundary (§2.6.1.4.1) is the origin, this conventional boundary does not include D which belongs to the relative interior in this dimension. (§2.7.0.0.1) $\square$

6.5.1 Gram-form correspondence to $\mathbb{S}_+^{N-1}$

With respect to $\mathbf{D}(G) = \delta(G)\mathbf{1}^{\mathrm T} + \mathbf{1}\delta(G)^{\mathrm T} - 2G$ (988) the linear Gram-form EDM operator, results in §5.6.1 provide [3, §2.6]

$$\mathbb{EDM}^N \;=\; \mathbf{D}\Big(\mathbf{V}(\mathbb{EDM}^N)\Big) \;\equiv\; \mathbf{D}\Big(V_{\mathcal N}\,\mathbb{S}_+^{N-1} V_{\mathcal N}^{\mathrm T}\Big) \tag{1309}$$

$$V_{\mathcal N}\,\mathbb{S}_+^{N-1} V_{\mathcal N}^{\mathrm T} \equiv \mathbf{V}\Big(\mathbf{D}\Big(V_{\mathcal N}\,\mathbb{S}_+^{N-1} V_{\mathcal N}^{\mathrm T}\Big)\Big) = \mathbf{V}(\mathbb{EDM}^N) \triangleq -V\,\mathbb{EDM}^N\, V\tfrac{1}{2} = \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1310}$$

a one-to-one correspondence between $\mathbb{EDM}^N$ and $\mathbb{S}_+^{N-1}$.

6.5.2 EDM cone by elliptope

Having defined the elliptope parametrized by scalar $t > 0$

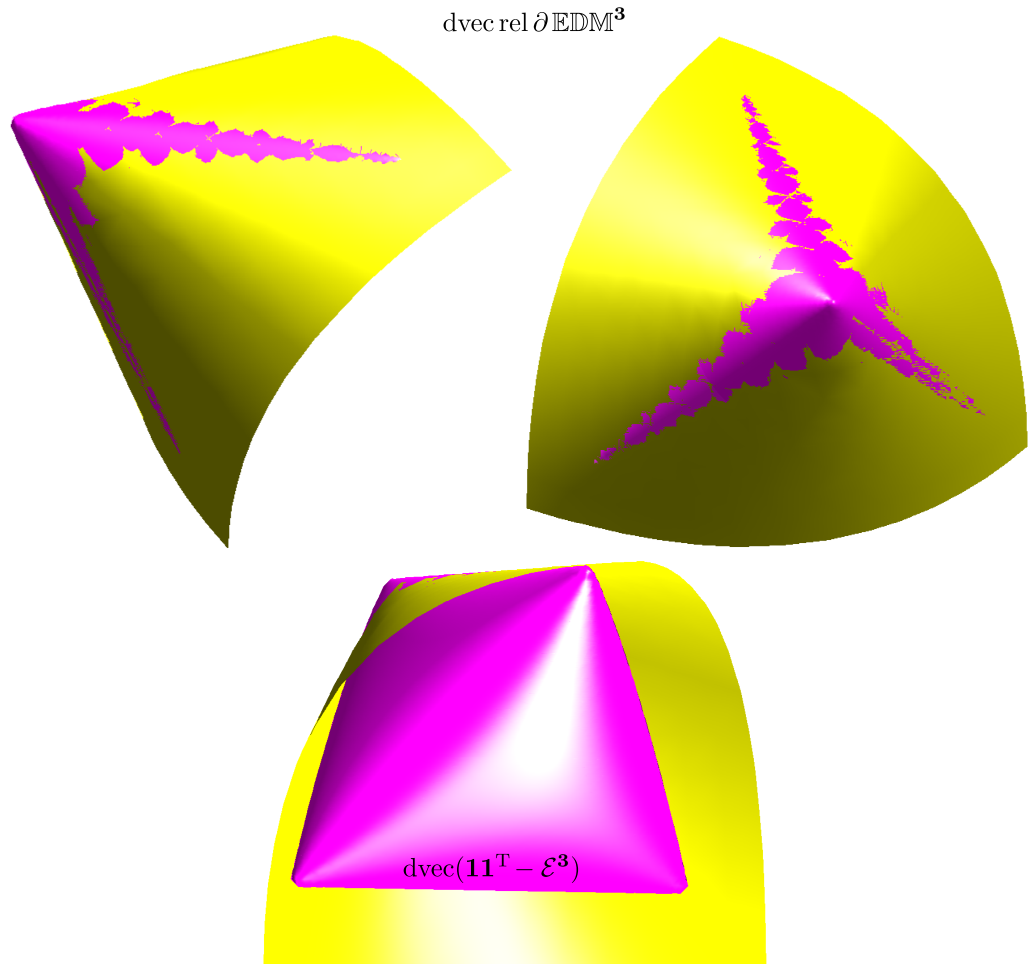
$$\mathcal{E}_t^N \;=\; \mathbb{S}_+^N \cap \{\Phi \in \mathbb{S}^N \mid \delta(\Phi) = t\mathbf{1}\} \tag{1185}$$

then following Alfakih [10] we have

$$\mathbb{EDM}^N \;=\; \overline{\operatorname{cone}\{\mathbf{1}\mathbf{1}^{\mathrm T} - \mathcal{E}_1^N\}} \;=\; \overline{\{t(\mathbf{1}\mathbf{1}^{\mathrm T} - \mathcal{E}_1^N) \mid t \geq 0\}} \tag{1311}$$

Identification $\mathcal{E}^N = \mathcal{E}_1^N$ equates the standard elliptope (§5.9.1.0.1, Figure **144**) to our parametrized elliptope.

[6.8] ($\Leftarrow$) For $a_i \in \mathbb{R}^{N-1}$, let $z_i = V_{\mathcal N}^{\dagger\mathrm T} a_i$.



$$\mathbb{EDM}^N = \overline{\operatorname{cone}\{\mathbf{11}^{\mathrm{T}} - \mathcal{E}^N\}} = \overline{\{t(\mathbf{11}^{\mathrm{T}} - \mathcal{E}^N) \mid t \geq 0\}} \tag{1311}$$

Figure 155: Three views of translated negated elliptope $\mathbf{11}^{\mathrm{T}} - \mathcal{E}_1^{\mathbf{3}}$ (*confer* Figure **144**) shrouded by truncated EDM cone. Fractal on EDM cone relative boundary is numerical artifact belonging to intersection with elliptope relative boundary. The fractal is trying to convey existence of a neighborhood about the origin where the translated elliptope boundary and EDM cone boundary intersect.

6.5.2.0.1 Expository. *Normal cone, tangent cone, elliptope.*
Define $\mathcal{T}_{\mathcal{E}}(\mathbf{1}\mathbf{1}^{\mathrm{T}})$ to be the *tangent cone* to the elliptope $\mathcal{E}$ at point $\mathbf{1}\mathbf{1}^{\mathrm{T}}$; *id est,*

$$\mathcal{T}_{\mathcal{E}}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) \triangleq \overline{\{t(\mathcal{E}-\mathbf{1}\mathbf{1}^{\mathrm{T}}) \mid t\geq 0\}} \tag{1312}$$

The normal cone $\mathcal{K}_{\mathcal{E}}^{\perp}(\mathbf{1}\mathbf{1}^{\mathrm{T}})$ to the elliptope at $\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is a closed convex cone defined (§E.10.3.2.1, Figure **191**)

$$\mathcal{K}_{\mathcal{E}}^{\perp}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) \triangleq \{B \mid \langle B\,,\, \Phi-\mathbf{1}\mathbf{1}^{\mathrm{T}}\rangle \leq 0\,,\ \Phi\in\mathcal{E}\} \tag{1313}$$

The *polar cone* of any set $\mathcal{K}$ is the closed convex cone (*confer* (296))

$$\mathcal{K}^{\circ} \triangleq \{B \mid \langle B\,,\, A\rangle\leq 0\,,\ \text{for all } A\in\mathcal{K}\} \tag{1314}$$

The normal cone is well known to be the polar of the tangent cone,

$$\mathcal{K}_{\mathcal{E}}^{\perp}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) = \mathcal{T}_{\mathcal{E}}(\mathbf{1}\mathbf{1}^{\mathrm{T}})^{\circ} \tag{1315}$$

and *vice versa*; [215, §A.5.2.4]

$$\mathcal{K}_{\mathcal{E}}^{\perp}(\mathbf{1}\mathbf{1}^{\mathrm{T}})^{\circ} = \mathcal{T}_{\mathcal{E}}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) \tag{1316}$$

From Deza & Laurent [120, p.535] we have the EDM cone

$$\mathbb{EDM} = -\mathcal{T}_{\mathcal{E}}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) \tag{1317}$$

The polar EDM cone is also expressible in terms of the elliptope. From (1315) we have

$$\mathbb{EDM}^{\circ} = -\mathcal{K}_{\mathcal{E}}^{\perp}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) \tag{1318}$$

★

In §5.10.1 we proposed the expression for EDM D

$$D = t\mathbf{1}\mathbf{1}^{\mathrm{T}} - \mathfrak{E} \in \mathbb{EDM}^N \tag{1186}$$

where $t\in\mathbb{R}_+$ and $\mathfrak{E}$ belongs to the parametrized elliptope $\mathcal{E}_t^N$. We further propose, for any particular $t>0$

$$\mathbb{EDM}^N = \overline{\operatorname{cone}\{t\mathbf{1}\mathbf{1}^{\mathrm{T}} - \mathcal{E}_t^N\}} \tag{1319}$$

Proof (pending). ■

6.5.2.0.2 Exercise. *EDM cone from elliptope.*
Relationship of the translated negated elliptope with the EDM cone is illustrated in Figure **155**. Prove whether it holds that

$$\mathbb{EDM}^N = \overline{\lim_{t\to\infty} t\mathbf{1}\mathbf{1}^{\mathrm{T}} - \mathcal{E}_t^N} \tag{1320}$$

▼

6.6 Vectorization & projection interpretation

In §E.7.2.0.2 we learn: $-VDV$ can be interpreted as orthogonal projection [7, §2] of vectorized $-D\in\mathbb{S}_h^N$ on the subspace of geometrically centered symmetric matrices

$$\begin{aligned}\mathbb{S}_c^N &= \{G\in\mathbb{S}^N \mid G\mathbf{1}=\mathbf{0}\}\\ &= \{G\in\mathbb{S}^N \mid \mathcal{N}(G)\supseteq\mathbf{1}\} = \{G\in\mathbb{S}^N \mid \mathcal{R}(G)\subseteq\mathcal{N}(\mathbf{1}^{\mathrm{T}})\}\\ &= \{VYV \mid Y\in\mathbb{S}^N\}\subset\mathbb{S}^N\\ &\equiv \{V_{\mathcal{N}}AV_{\mathcal{N}}^{\mathrm{T}} \mid A\in\mathbb{S}^{N-1}\}\end{aligned} \tag{1078}$$

because elementary auxiliary matrix V is an orthogonal projector (§B.4.1). Yet there is another useful projection interpretation:

Revising the fundamental matrix criterion for membership to the EDM cone (971),[6.9]

$$\left.\begin{array}{r}\langle zz^{\mathrm{T}},-D\rangle\geq 0 \quad \forall\, zz^{\mathrm{T}} \mid \mathbf{1}\mathbf{1}^{\mathrm{T}}zz^{\mathrm{T}}=\mathbf{0}\\ D\in\mathbb{S}_h^N\end{array}\right\} \Leftrightarrow D\in\mathbb{EDM}^N \tag{1321}$$

this is equivalent, of course, to the Schoenberg criterion

$$\left.\begin{array}{r}-V_{\mathcal{N}}^{\mathrm{T}}DV_{\mathcal{N}}\succeq 0\\ D\in\mathbb{S}_h^N\end{array}\right\} \Leftrightarrow D\in\mathbb{EDM}^N \tag{995}$$

because $\mathcal{N}(\mathbf{1}\mathbf{1}^{\mathrm{T}})=\mathcal{R}(V_{\mathcal{N}})$. When $D\in\mathbb{EDM}^N$, correspondence (1321) means $-z^{\mathrm{T}}Dz$ is proportional to a nonnegative coefficient of orthogonal projection (§E.6.4.2, Figure **157**) of $-D$ in isometrically isomorphic $\mathbb{R}^{N(N+1)/2}$ on the range of each and every vectorized (§2.2.2.1) symmetric dyad (§B.1) in the nullspace of $\mathbf{1}\mathbf{1}^{\mathrm{T}}$; *id est*, on each and every member of

$$\begin{aligned}\mathcal{T} &\triangleq \left\{\operatorname{svec}(zz^{\mathrm{T}}) \mid z\in\mathcal{N}(\mathbf{1}\mathbf{1}^{\mathrm{T}})=\mathcal{R}(V_{\mathcal{N}})\right\}\subset \operatorname{svec}\partial\mathbb{S}_+^N\\ &= \left\{\operatorname{svec}(V_{\mathcal{N}}\upsilon\upsilon^{\mathrm{T}}V_{\mathcal{N}}^{\mathrm{T}}) \mid \upsilon\in\mathbb{R}^{N-1}\right\}\end{aligned} \tag{1322}$$

whose dimension is

$$\dim\mathcal{T} = N(N-1)/2 \tag{1323}$$

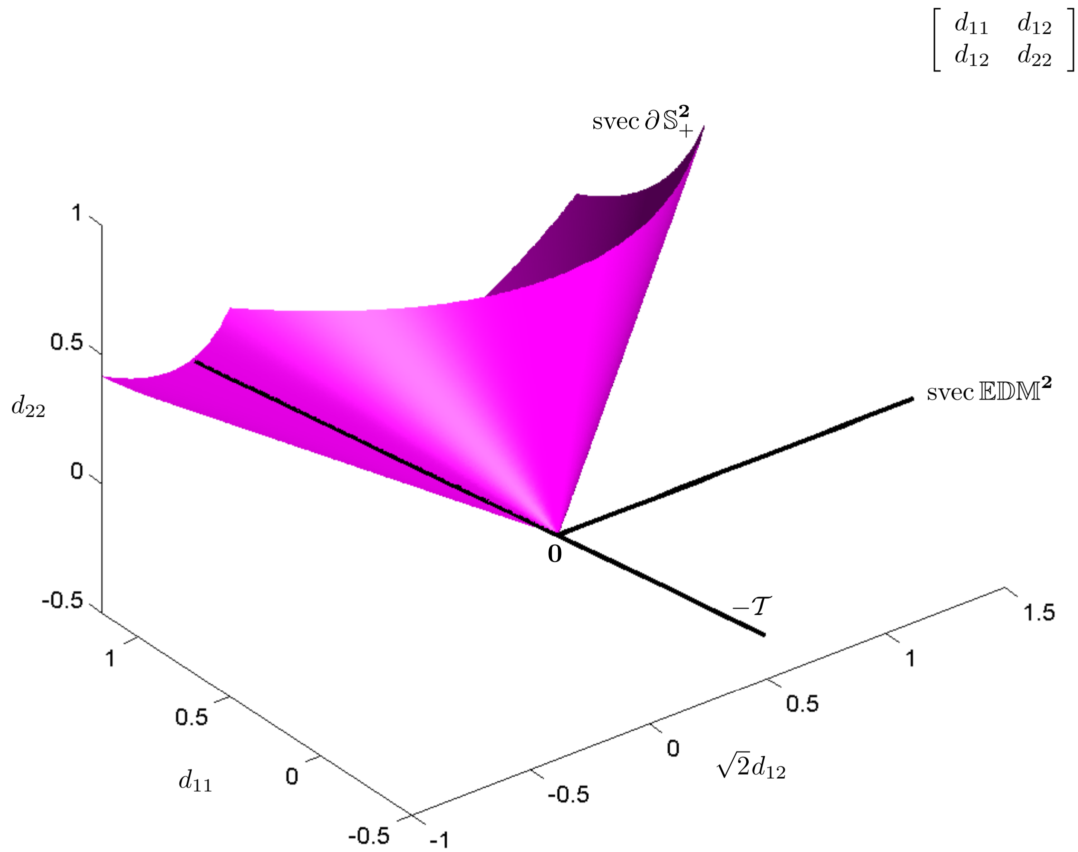
The set of all symmetric dyads $\{zz^{\mathrm{T}} \mid z\in\mathbb{R}^N\}$ constitute the extreme directions of the positive semidefinite cone (§2.8.1, §2.9) $\mathbb{S}_+^N$, hence lie on its boundary. Yet only those dyads in $\mathcal{R}(V_{\mathcal{N}})$ are included in the test (1321), thus only a subset $\mathcal{T}$ of all vectorized extreme directions of $\mathbb{S}_+^N$ is observed.

In the particularly simple case $D\in\mathbb{EDM}^{\mathbf{2}}=\{D\in\mathbb{S}_h^{\mathbf{2}} \mid d_{12}\geq 0\}$, for example, only one extreme direction of the PSD cone is involved:

$$zz^{\mathrm{T}} = \begin{bmatrix} 1 & -1\\ -1 & 1\end{bmatrix} \tag{1324}$$

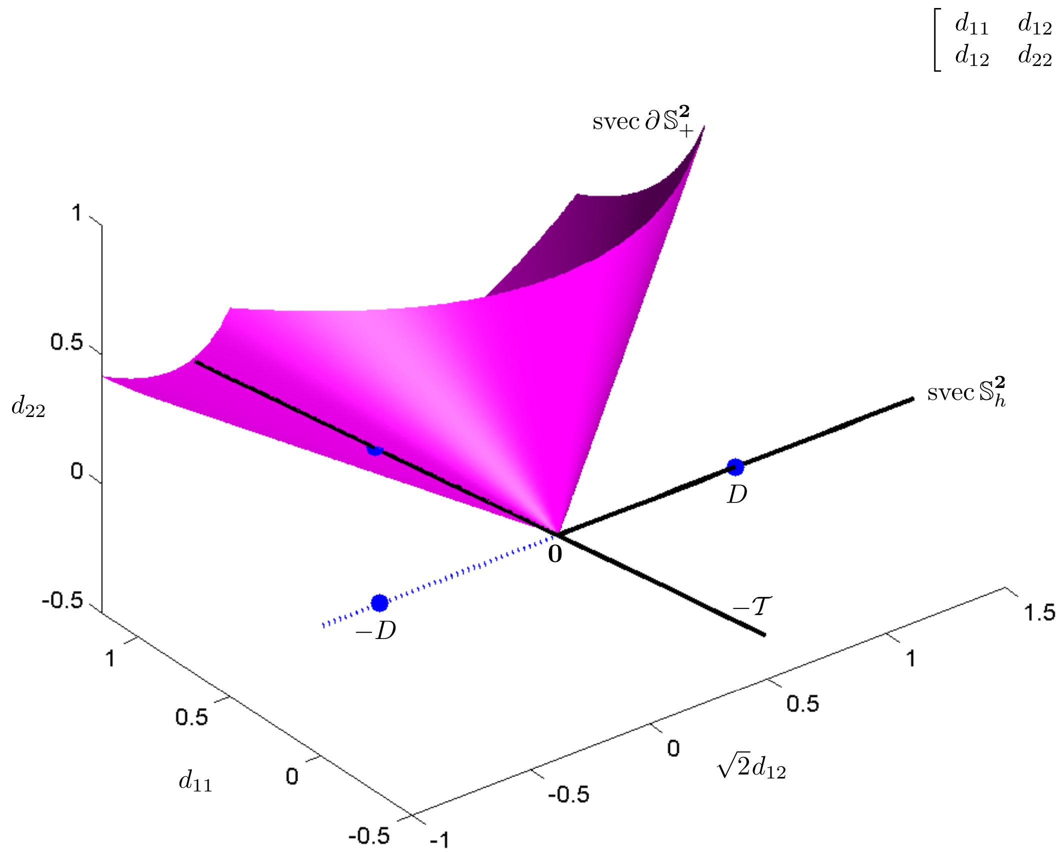
Any nonnegative scaling of vectorized zz^{T} belongs to the set $\mathcal{T}$ illustrated in Figure **156** and Figure **157**.

[6.9] $\mathcal{N}(\mathbf{1}\mathbf{1}^{\mathrm{T}})=\mathcal{N}(\mathbf{1}^{\mathrm{T}})$ and $\mathcal{R}(zz^{\mathrm{T}})=\mathcal{R}(z)$



$$\mathcal{T} \triangleq \left\{ \text{svec}(zz^{\mathrm{T}}) \mid z \in \mathcal{N}(\mathbf{1}\mathbf{1}^{\mathrm{T}}) = \kappa \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \kappa \in \mathbb{R} \right\} \subset \text{svec}\,\partial\mathbb{S}^{\mathbf{2}}_+$$

Figure 156: Truncated boundary of positive semidefinite cone $\mathbb{S}^{\mathbf{2}}_+$ in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$ (via svec (56)) is, in this dimension, constituted solely by its extreme directions. Truncated cone of Euclidean distance matrices $\mathbb{EDM}^{\mathbf{2}}$ in isometrically isomorphic subspace $\mathbb{R}$. Relative boundary of EDM cone is constituted solely by matrix $\mathbf{0}$. Halfline $\mathcal{T} = \{\kappa^2[1\ \ -\sqrt{2}\ \ \ 1]^{\mathrm{T}} \mid \kappa \in \mathbb{R}\}$ on PSD cone boundary depicts that lone extreme ray (1324) on which orthogonal projection of $-D$ must be positive semidefinite if D is to belong to $\mathbb{EDM}^{\mathbf{2}}$. aff cone $\mathcal{T} = \text{svec}\,\mathbb{S}^{\mathbf{2}}_c$. (1329) Dual EDM cone is halfspace in $\mathbb{R}^{\mathbf{3}}$ whose bounding hyperplane has inward-normal $\text{svec}\,\mathbb{EDM}^{\mathbf{2}}$.



$P_{\operatorname{svec} zz^{\mathrm{T}}}(\operatorname{svec}(-D)) = \dfrac{\langle zz^{\mathrm{T}} ,\, -D\rangle}{\langle zz^{\mathrm{T}} ,\, zz^{\mathrm{T}}\rangle}\, zz^{\mathrm{T}}$ is projection of vectorized $-D$ on range of vectorized zz^{T}.

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; \begin{cases} \langle zz^{\mathrm{T}} , -D\rangle \geq 0 \quad \forall\, zz^{\mathrm{T}} \mid \mathbf{1}\mathbf{1}^{\mathrm{T}} zz^{\mathrm{T}} = \mathbf{0} \\ D \in \mathbb{S}^N_h \end{cases} \tag{1321}$$

Figure 157: Given-matrix D is assumed to belong to symmetric hollow subspace $\mathbb{S}^{\mathbf{2}}_h$; a line in this dimension. Negative D is found along $\mathbb{S}^{\mathbf{2}}_h$. Set $\mathcal{T}$ (1322) has only one ray member in this dimension; not orthogonal to $\mathbb{S}^{\mathbf{2}}_h$. Orthogonal projection of $-D$ on $\mathcal{T}$ (indicated by half dot) has nonnegative projection coefficient. Matrix D must therefore be an EDM.

6.6.1 Face of PSD cone $\mathbb{S}_+^N$ containing V

In any case, set $\mathcal{T}$ (1322) constitutes the vectorized extreme directions of an $N(N-1)/2$-dimensional face of the PSD cone $\mathbb{S}_+^N$ containing auxiliary matrix V; a face isomorphic with $\mathbb{S}_+^{N-1} = \mathbb{S}_+^{\operatorname{rank} V}$ (§2.9.2.3).

To show this, we must first find the smallest face that contains auxiliary matrix V and then determine its extreme directions. From (221),

$$\begin{aligned}\mathcal{F}\Big(\mathbb{S}_+^N \ni V\Big) \;&=\; \{W \in \mathbb{S}_+^N \mid \mathcal{N}(W) \supseteq \mathcal{N}(V)\} \;=\; \{W \in \mathbb{S}_+^N \mid \mathcal{N}(W) \supseteq \mathbf{1}\}\\ &=\; \{VYV \succeq 0 \mid Y \in \mathbb{S}^N\} \;\equiv\; \{V_{\mathcal{N}} B V_{\mathcal{N}}^{\mathrm{T}} \mid B \in \mathbb{S}_+^{N-1}\}\\ &\simeq\; \mathbb{S}_+^{\operatorname{rank} V} \;=\; -V_{\mathcal{N}}^{\mathrm{T}}\, \mathbb{EDM}^N V_{\mathcal{N}}\end{aligned} \tag{1325}$$

where the equivalence $\equiv$ is from §5.6.1 while isomorphic equality $\simeq$ with transformed EDM cone is from (1108). Projector V belongs to $\mathcal{F}(\mathbb{S}_+^N \ni V)$ because $V_{\mathcal{N}} V_{\mathcal{N}}^{\dagger} V_{\mathcal{N}}^{\dagger\mathrm{T}} V_{\mathcal{N}}^{\mathrm{T}} = V$ (§B.4.3). Each and every rank-one matrix belonging to this face is therefore of the form:

$$V_{\mathcal{N}} \upsilon \upsilon^{\mathrm{T}} V_{\mathcal{N}}^{\mathrm{T}} \mid \upsilon \in \mathbb{R}^{N-1} \tag{1326}$$

Because $\mathcal{F}(\mathbb{S}_+^N \ni V)$ is isomorphic with a positive semidefinite cone $\mathbb{S}_+^{N-1}$, then $\mathcal{T}$ constitutes the vectorized extreme directions of $\mathcal{F}$, the origin constitutes the extreme points of $\mathcal{F}$, and auxiliary matrix V is some convex combination of those extreme points and directions by the *extremes theorem* (§2.8.1.1.1).

♦

In fact the smallest face, that contains auxiliary matrix V, of the PSD cone $\mathbb{S}_+^N$ is the intersection with the geometric center subspace (2113) (2114);

$$\begin{aligned}\mathcal{F}\Big(\mathbb{S}_+^N \ni V\Big) \;&=\; \operatorname{cone}\Big\{V_{\mathcal{N}} \upsilon \upsilon^{\mathrm{T}} V_{\mathcal{N}}^{\mathrm{T}} \mid \upsilon \in \mathbb{R}^{N-1}\Big\}\\ &=\; \mathbb{S}_c^N \cap \mathbb{S}_+^N\\ &\equiv \{X \succeq 0 \mid \langle X\,,\, \mathbf{1}\mathbf{1}^{\mathrm{T}}\rangle = 0\} \qquad (1684)\end{aligned} \tag{1327}$$

In isometrically isomorphic $\mathbb{R}^{N(N+1)/2}$

$$\operatorname{svec} \mathcal{F}\Big(\mathbb{S}_+^N \ni V\Big) \;=\; \operatorname{cone} \mathcal{T} \tag{1328}$$

related to $\mathbb{S}_c^N$ by

$$\operatorname{aff} \operatorname{cone} \mathcal{T} \;=\; \operatorname{svec} \mathbb{S}_c^N \tag{1329}$$

6.6.2 EDM criteria in $\mathbf{1}\mathbf{1}^{\mathrm{T}}$

(*confer* §6.4, (1002)) Laurent specifies an elliptope trajectory condition for EDM cone membership: [251, §2.3]

$$D \in \mathbb{EDM}^N \;\Leftrightarrow\; [1 - e^{-\alpha\, d_{ij}}] \in \mathbb{EDM}^N \quad \forall\, \alpha > 0 \qquad (1180\text{a})$$

From the parametrized elliptope $\mathcal{E}_t^N$ in §6.5.2 and §5.10.1 we propose

$$D \in \mathbb{EDM}^N \quad \Leftrightarrow \quad \exists \begin{array}{c} t \in \mathbb{R}_+ \\ \mathfrak{E} \in \mathcal{E}_t^N \end{array} \Bigg\} \ni \ D = t\mathbf{1}\mathbf{1}^{\mathrm{T}} - \mathfrak{E} \tag{1330}$$

Chabrillac & Crouzeix [77, §4] prove a different criterion they attribute to Finsler, 1937 [156]. We apply it to EDMs: for $D \in \mathbb{S}_h^N$ (1128)

$$\begin{array}{c} -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \succ 0 \quad \Leftrightarrow \quad \exists\, \kappa > 0 \ \ni \ -D + \kappa \mathbf{1}\mathbf{1}^{\mathrm{T}} \succ 0 \\ \Leftrightarrow \\ D \in \mathbb{EDM}^N \text{ with corresponding affine dimension } r = N-1 \end{array} \tag{1331}$$

This *Finsler criterion* has geometric interpretation in terms of the vectorization & projection already discussed in connection with (1321). With reference to Figure **156**, the offset $\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is simply a direction orthogonal to $\mathcal{T}$ in isomorphic $\mathbb{R}^{\mathbf{3}}$. Intuitively, translation of $-D$ in direction $\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is like orthogonal projection on $\mathcal{T}$ insofar as similar information can be obtained.

When the Finsler criterion (1331) is applied despite lower affine dimension, the constant κ can go to infinity making the test $-D + \kappa \mathbf{1}\mathbf{1}^{\mathrm{T}} \succeq 0$ impractical for numerical computation. Chabrillac & Crouzeix invent a criterion for the semidefinite case, but is no more practical: for $D \in \mathbb{S}_h^N$

$$\begin{array}{c} D \in \mathbb{EDM}^N \\ \Leftrightarrow \\ \exists\, \kappa_{\mathrm{p}} > 0 \ \ni \ \forall\, \kappa \geq \kappa_{\mathrm{p}} \ , \ -D - \kappa \mathbf{1}\mathbf{1}^{\mathrm{T}} \ [\textit{sic}] \text{ has exactly one negative eigenvalue} \end{array} \tag{1332}$$

6.7 A geometry of completion

It is not known how to proceed if one wishes to restrict the dimension of the Euclidean space in which the configuration of points may be constructed.

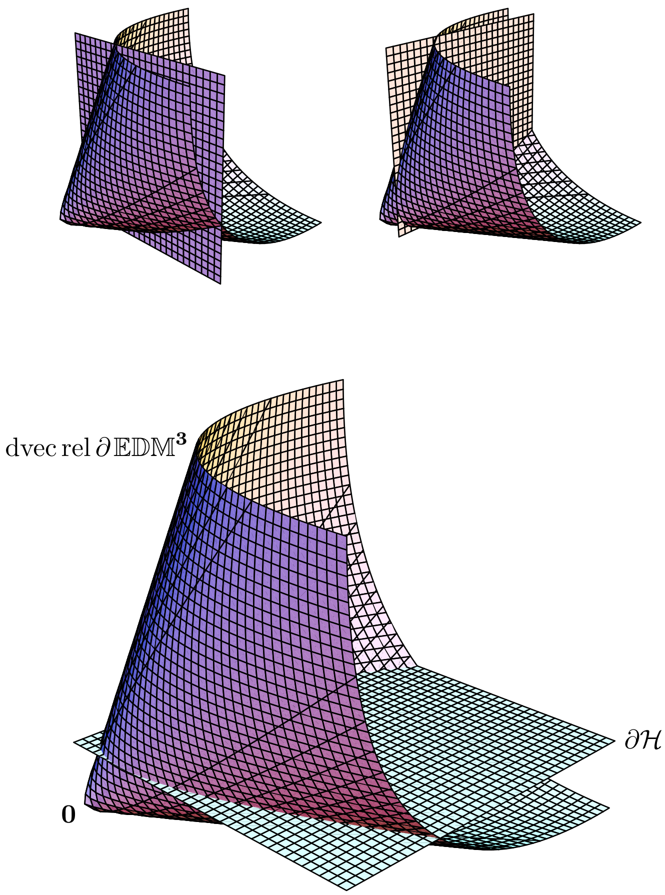
−Michael W. Trosset, 2000 [371, §1]

Given an incomplete noiseless EDM, intriguing is the question of whether a list in $X \in \mathbb{R}^{n \times N}$ (76) may be reconstructed and under what circumstances reconstruction is unique. [3] [5] [6] [7] [9] [18] [70] [221] [233] [250] [251] [252]

If one or more entries of a particular EDM are fixed, then geometric interpretation of the feasible set of completions is the intersection of the EDM cone $\mathbb{EDM}^N$ in isomorphic subspace $\mathbb{R}^{N(N-1)/2}$ with as many hyperplanes as there are fixed symmetric entries.[6.10] Assuming a nonempty intersection, then the number of completions is generally infinite, and those corresponding to particular affine dimension $r < N-1$ belong to some generally nonconvex subset of that intersection (*confer* §2.9.2.9.2) that can be as small as a point.

[6.10] Depicted in Figure **158**a is an intersection of the EDM cone $\mathbb{EDM}^{\mathbf{3}}$ with a single hyperplane representing the set of all EDMs having one fixed symmetric entry. This representation is practical because it is easily combined with or replaced by other convex constraints; *e.g*, slab inequalities in (789) that admit bounding of noise processes.

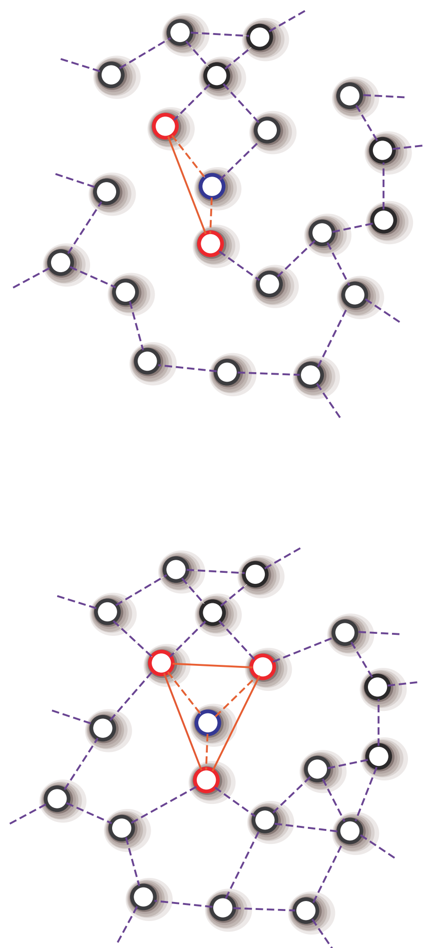
(b) (c)



(a)

Figure 158: **(a)** In isometrically isomorphic subspace $\mathbb{R}^3$, intersection of $\mathbb{EDM}^3$ with hyperplane $\partial\mathcal{H}$ representing one fixed symmetric entry $d_{23}=\kappa$ (both drawn truncated, rounded vertex is artifact of plot). EDMs in this dimension corresponding to affine dimension 1 comprise relative boundary of EDM cone (§6.5). Since intersection illustrated includes a nontrivial subset of cone's relative boundary, then it is apparent there exist infinitely many EDM completions corresponding to affine dimension 1. In this dimension it is impossible to represent a unique nonzero completion corresponding to affine dimension 1, for example, using a single hyperplane because any hyperplane supporting relative boundary at a particular point Γ contains an entire ray $\{\zeta\Gamma \mid \zeta\geq 0\}$ belonging to $\operatorname{rel}\partial\mathbb{EDM}^3$ by Lemma 2.8.0.0.1. **(b)** $d_{13}=\kappa$. **(c)** $d_{12}=\kappa$.

(a) two nearest neighbors



(b) three nearest neighbors

Figure 159: One dimensionless EDM subgraph completion (solid) superimposed on (but not obscuring) neighborhood graph (dashed). Local view of a few dense samples ○ from relative interior of some arbitrary Euclidean manifold whose affine dimension appears two-dimensional in this neighborhood. All line segments measure absolute distance. Dashed line segments help visually locate nearest neighbors; suggesting, best number of nearest neighbors can be greater than value of embedding dimension after topological transformation (*confer* [228, §2]). Solid line segments represent completion of EDM subgraph from available distance data for an arbitrarily chosen sample and its nearest neighbors. Each distance from EDM subgraph becomes distance-square in corresponding EDM submatrix.

6.7.0.0.1 Example. *Maximum variance unfolding.* [411]
A process minimizing affine dimension (§2.1.5) of certain kinds of Euclidean *manifold* by topological transformation can be posed as a completion problem (*confer* §E.10.2.1.2). Weinberger & Saul, who originated the technique, specify an applicable manifold in three dimensions by analogy to an ordinary sheet of paper (*confer* §2.1.6); imagine, we find it deformed from flatness in some way introducing neither holes, tears, or selfintersections. [411, §2.2] The physical process is intuitively described as *unfurling, unfolding, diffusing, decompacting,* or *unraveling.* In particular instances, the process is a sort of flattening by stretching until taut (but not by crushing); *e.g*, unfurling a three-dimensional Euclidean body resembling a billowy national flag reduces that manifold's affine dimension to $r=2$.

Data input to the proposed process originates from distances between neighboring relatively dense samples of a given manifold. Figure **159** realizes a densely sampled neighborhood; called, *neighborhood graph.* Essentially, the algorithmic process preserves local isometry between *nearest neighbors* allowing distant neighbors to excurse expansively by "maximizing variance" (Figure **7**). The common number of nearest neighbors to each sample is a data-dependent algorithmic parameter whose minimum value connects the graph. The dimensionless *EDM subgraph* between each sample and its nearest neighbors is completed from available data and included as input; one such EDM subgraph completion is drawn superimposed upon the neighborhood graph in Figure **159**.[6.11] The consequent dimensionless EDM graph comprising all the subgraphs is incomplete, in general, because the neighbor number is relatively small; incomplete even though it is a superset of the neighborhood graph. Remaining distances (those not graphed at all) are squared then made variables within the algorithm; it is this variability that admits unfurling.

To demonstrate, consider untying the *trefoil knot* drawn in Figure **160**a. A corresponding EDM $D=[d_{ij}\ ,\ i,j=1\dots N]$ employing only two nearest neighbors is banded having the incomplete form

$$D = \begin{bmatrix} 0 & \check{d}_{12} & \check{d}_{13} & ? & \cdots & ? & \check{d}_{1,N-1} & \check{d}_{1N} \\ \check{d}_{12} & 0 & \check{d}_{23} & \check{d}_{24} & \ddots & ? & ? & \check{d}_{2N} \\ \check{d}_{13} & \check{d}_{23} & 0 & \check{d}_{34} & \ddots & ? & ? & ? \\ ? & \check{d}_{24} & \check{d}_{34} & 0 & \ddots & \ddots & ? & ? \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & ? \\ ? & ? & ? & \ddots & \ddots & 0 & \check{d}_{N-2,N-1} & \check{d}_{N-2,N} \\ \check{d}_{1,N-1} & ? & ? & ? & \ddots & \check{d}_{N-2,N-1} & 0 & \check{d}_{N-1,N} \\ \check{d}_{1N} & \check{d}_{2N} & ? & ? & ? & \check{d}_{N-2,N} & \check{d}_{N-1,N} & 0 \end{bmatrix} \tag{1333}$$

where $\check{d}_{ij}$ denotes a given fixed distance-square. The unfurling algorithm can be expressed

[6.11] Local reconstruction of point position, from the EDM submatrix corresponding to a complete dimensionless EDM subgraph, is unique to within an isometry (§5.6, §5.12).

(a)

(b)

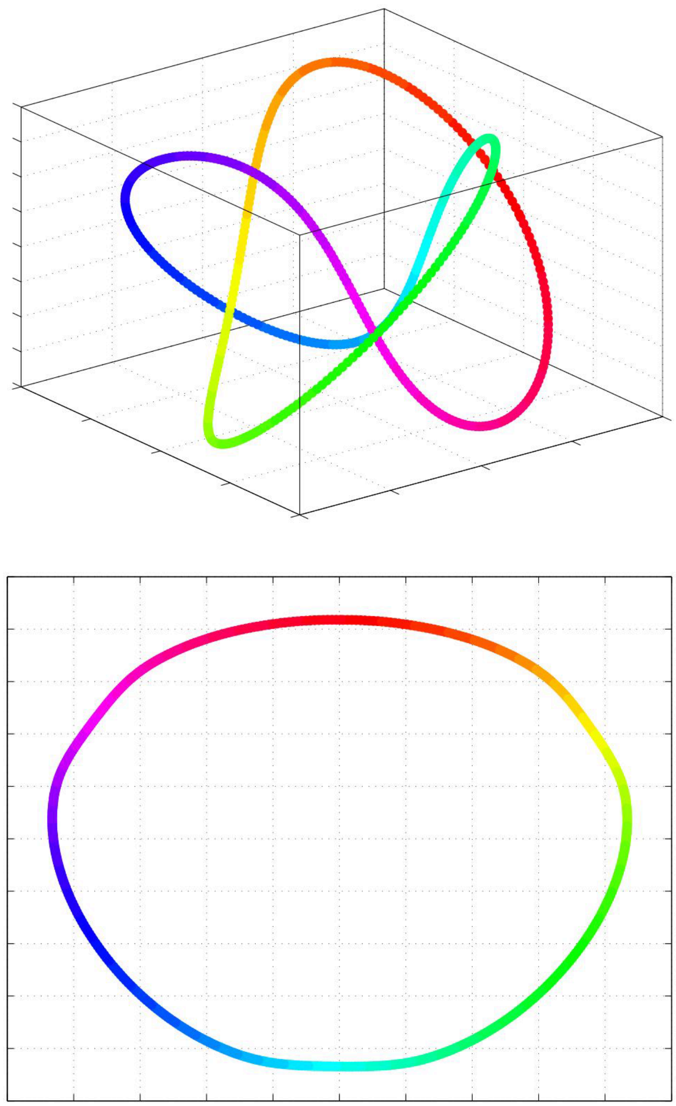


Figure 160: **(a)** *Trefoil knot* in $\mathbb{R}^3$ from Weinberger & Saul [411]. **(b)** Topological transformation algorithm employing four nearest neighbors and $N = 539$ samples reduces affine dimension of knot to $r = 2$. Choosing instead two nearest neighbors would make this embedding more circular.

as an optimization problem; constrained total distance-square maximization:

$$\begin{array}{ll}\underset{D}{\text{maximize}} & \langle -V\,,\,D\rangle\\ \text{subject to} & \langle D\,,\,e_i e_j^{\mathrm T} + e_j e_i^{\mathrm T}\rangle\frac{1}{2} = \check{d}_{ij} \qquad \forall(i\,,j)\in\mathcal{I}\\ & \operatorname{rank}(VDV) = 2\\ & D\in\mathbb{EDM}^N\end{array} \tag{1334}$$

where $e_i\in\mathbb{R}^N$ is the i^{th} member of the standard basis, where set $\mathcal{I}$ indexes the given distance-square data like that in (1333), where $V\in\mathbb{R}^{N\times N}$ is the geometric centering matrix (§B.4.1), and where

$$\langle -V\,,\,D\rangle \;=\; \operatorname{tr}(-VDV) \;=\; 2\operatorname{tr}G \;=\; \frac{1}{N}\sum_{i,j} d_{ij} \tag{1000}$$

where G is the Gram matrix producing D assuming $G\mathbf{1}=\mathbf{0}$.

If the (rank) constraint on affine dimension is ignored, then problem (1334) becomes convex, a corresponding solution $D^\star$ can be found, and a nearest rank-2 solution is then had by ordered eigenvalue decomposition of $-VD^\star V$ followed by *spectral projection* (§7.1.3) on $\left[\begin{array}{c}\mathbb{R}^{\mathbf{2}}_+\\ \mathbf{0}\end{array}\right]\subset\mathbb{R}^N$. This two-step process is necessarily suboptimal. Yet because the decomposition for the trefoil knot reveals only two dominant eigenvalues, the spectral projection is nearly benign. Such a reconstruction of point position (§5.12) utilizing four nearest neighbors is drawn in Figure **160**b; a low-dimensional embedding of the trefoil knot.

This problem (1334) can, of course, be written equivalently in terms of Gram matrix G, facilitated by (1006); *videlicet*, for Φ_{ij} as in (974)

$$\begin{array}{ll}\underset{G\in\mathbb{S}_c^N}{\text{maximize}} & \langle I\,,\,G\rangle\\ \text{subject to} & \langle G\,,\,\Phi_{ij}\rangle = \check{d}_{ij} \qquad \forall(i\,,j)\in\mathcal{I}\\ & \operatorname{rank}G = 2\\ & G\succeq 0\end{array} \tag{1335}$$

The advantage to converting EDM to Gram is: Gram matrix G is a bridge between point list X and EDM D; constraints on any or all of these three variables may now be introduced. (Example 5.4.2.2.8) Confinement to the geometric center subspace $\mathbb{S}_c^N$ (implicit constraint $G\mathbf{1}=\mathbf{0}$) keeps G independent of its translation-invariant subspace $\mathbb{S}_c^{N\perp}$ (§5.5.1.1, Figure **162**) so as not to become numerically unbounded.

A problem dual to *maximum variance unfolding problem* (1335) (less the Gram rank constraint) has been called the *fastest mixing Markov process.* That dual has simple interpretations in graph and circuit theory and in mechanical and thermal systems, explored in [357], and has direct application to quick calculation of *PageRank* by search engines [247, §4]. Optimal Gram rank turns out to be tightly bounded above by minimum multiplicity of the second smallest eigenvalue of a dual optimal variable. □

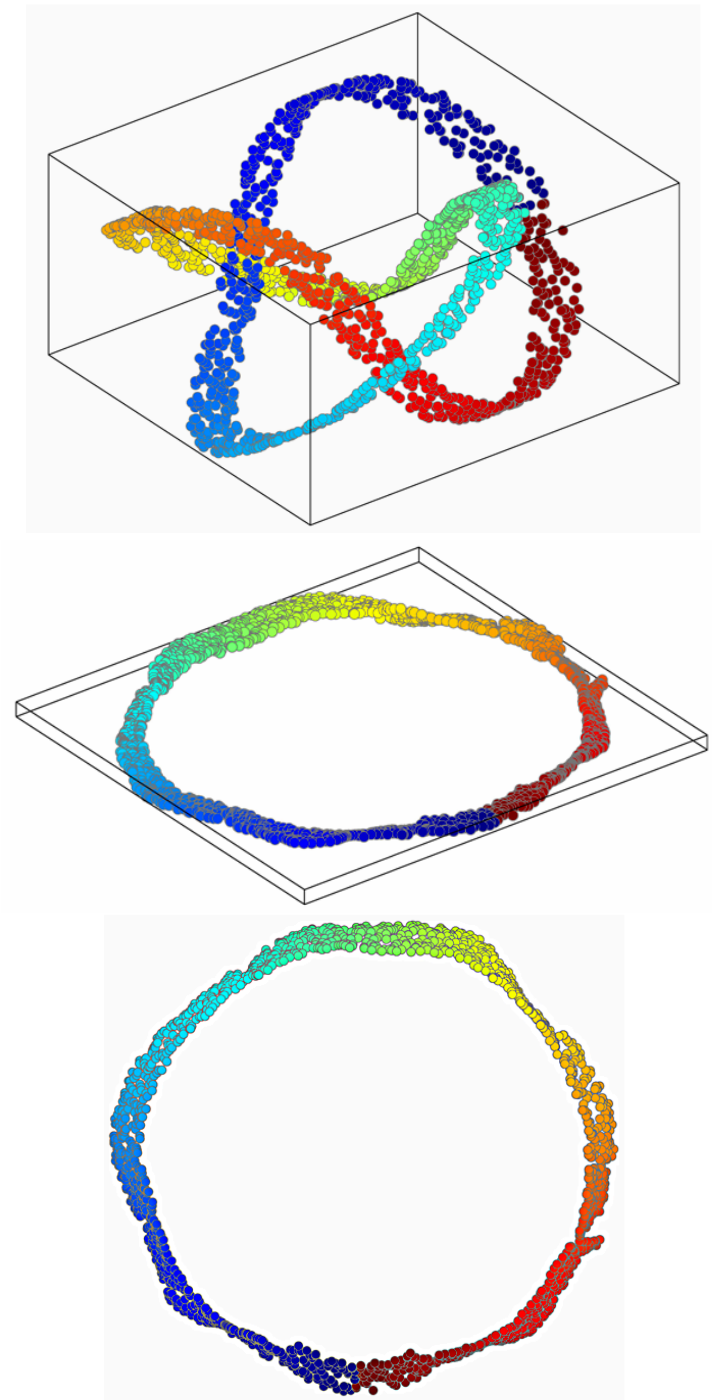


Figure 161: *Trefoil ribbon* (Kilian Weinberger). Same topological transformation algorithm as in Figure 160b with five nearest neighbors and $N = 1617$ samples.

6.8 Dual EDM cone

6.8.1 Ambient $\mathbb{S}^N$

We consider finding the ordinary dual EDM cone in ambient space $\mathbb{S}^N$ where $\mathbb{EDM}^N$ is pointed, closed, convex, but not full-dimensional. The set of all EDMs in $\mathbb{S}^N$ is a closed convex cone because it is the intersection of halfspaces about the origin in vectorized variable D (§2.4.1.1.1, §2.7.2):

$$\mathbb{EDM}^N = \bigcap_{\substack{z\in\mathcal{N}(\mathbf{1}^{\mathrm T})\\ i=1\ldots N}} \left\{ D\in\mathbb{S}^N \mid \langle e_i e_i^{\mathrm T}, D\rangle \geq 0\,,\ \langle e_i e_i^{\mathrm T}, D\rangle \leq 0\,,\ \langle zz^{\mathrm T}, -D\rangle \geq 0 \right\} \tag{1336}$$

By definition (296), dual cone $\mathcal{K}^*$ comprises each and every vector inward-normal to a hyperplane supporting convex cone $\mathcal{K}$. The dual EDM cone in the ambient space of symmetric matrices is therefore expressible as the aggregate of every conic combination of inward-normals from (1336):

$$\begin{aligned}
\mathbb{EDM}^{N^*} &= \operatorname{cone}\{e_i e_i^{\mathrm T}\,,\ -e_j e_j^{\mathrm T} \mid i\,,j=1\ldots N\} \ -\ \operatorname{cone}\{zz^{\mathrm T} \mid \mathbf{11}^{\mathrm T}zz^{\mathrm T}=\mathbf{0}\}\\
&= \{\sum_{i=1}^N \zeta_i\, e_i e_i^{\mathrm T} - \sum_{j=1}^N \xi_j\, e_j e_j^{\mathrm T} \mid \zeta_i\,,\xi_j \geq 0\} \ -\ \operatorname{cone}\{zz^{\mathrm T} \mid \mathbf{11}^{\mathrm T}zz^{\mathrm T}=\mathbf{0}\}\\
&= \{\delta(u) \mid u\in\mathbb{R}^N\} - \operatorname{cone}\left\{V_{\mathcal{N}}\upsilon\upsilon^{\mathrm T}V_{\mathcal{N}}^{\mathrm T} \mid \upsilon\in\mathbb{R}^{N-1},\ (\|\upsilon\|=1)\right\} \subset \mathbb{S}^N\\
&= \{\delta^2(Y) - V_{\mathcal{N}}\Psi V_{\mathcal{N}}^{\mathrm T} \mid Y\in\mathbb{S}^N,\ \Psi\in\mathbb{S}_+^{N-1}\}
\end{aligned} \tag{1337}$$

The EDM cone is not selfdual in ambient $\mathbb{S}^N$ because its affine hull belongs to a proper subspace

$$\operatorname{aff}\mathbb{EDM}^N = \mathbb{S}_h^N \tag{1338}$$

The ordinary dual EDM cone cannot, therefore, be pointed. (§2.13.1.1)

When $N=1$, the EDM cone is the point at the origin in $\mathbb{R}$. Auxiliary matrix $V_{\mathcal{N}}$ is empty $[\,\emptyset\,]$, and dual cone $\mathbb{EDM}^*$ is the real line.

When $N=2$, the EDM cone is a nonnegative real line in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$; there $\mathbb{S}_h^{\mathbf{2}}$ is a real line containing the EDM cone. Dual cone $\mathbb{EDM}^{\mathbf{2}^*}$ is the particular halfspace in $\mathbb{R}^{\mathbf{3}}$ whose boundary has inward-normal $\mathbb{EDM}^{\mathbf{2}}$. Diagonal matrices $\{\delta(u)\}$ in (1337) are represented by a hyperplane through the origin $\{\underline{d} \mid [\,0\ \ 1\ \ 0\,]\underline{d}=0\}$ while the term $\operatorname{cone}\{V_{\mathcal{N}}\upsilon\upsilon^{\mathrm T}V_{\mathcal{N}}^{\mathrm T}\}$ is represented by the halfline $\mathcal{T}$ in Figure **156** belonging to the positive semidefinite cone boundary. The dual EDM cone is formed by translating the hyperplane along the negative semidefinite halfline $-\mathcal{T}$; the union of each and every translation. (*confer* §2.10.2.0.1)

When cardinality N exceeds 2, the dual EDM cone can no longer be polyhedral simply because the EDM cone cannot. (§2.13.1.1)

6.8.1.1 EDM cone and its dual in ambient $\mathbb{S}^N$

Consider the two convex cones

$$\begin{aligned}\mathcal{K}_1 &\triangleq \mathbb{S}_h^N \\ \mathcal{K}_2 &\triangleq \bigcap_{y\in\mathcal{N}(\mathbf{1}^{\mathrm{T}})}\left\{A\in\mathbb{S}^N \mid \langle yy^{\mathrm{T}}, -A\rangle \geq 0\right\} \\ &= \left\{A\in\mathbb{S}^N \mid -z^{\mathrm{T}}VA\,Vz\geq 0 \quad \forall\, zz^{\mathrm{T}}(\succeq 0)\right\} \\ &= \left\{A\in\mathbb{S}^N \mid -VA\,V\succeq 0\right\}\end{aligned} \tag{1339}$$

so

$$\mathcal{K}_1\cap\mathcal{K}_2 = \mathbb{EDM}^N \tag{1340}$$

Dual cone $\mathcal{K}_1^* = \mathbb{S}_h^{N\perp}\subseteq\mathbb{S}^N$ (72) is the subspace of diagonal matrices. From (1337) via (313),

$$\mathcal{K}_2^* = -\operatorname{cone}\left\{V_{\mathcal{N}}\upsilon\upsilon^{\mathrm{T}}V_{\mathcal{N}}^{\mathrm{T}} \mid \upsilon\in\mathbb{R}^{N-1}\right\}\subset\mathbb{S}^N \tag{1341}$$

Gaffke & Mathar [161, §5.3] observe that projection on $\mathcal{K}_1$ and $\mathcal{K}_2$ have simple closed forms: Projection on subspace $\mathcal{K}_1$ is easily performed by symmetrization and zeroing the main diagonal or *vice versa,* while projection of $H\in\mathbb{S}^N$ on $\mathcal{K}_2$ is[6.12]

$$P_{\mathcal{K}_2}H = H - P_{\mathbb{S}_+^N}(VH\,V) \tag{1342}$$

Proof. First, we observe membership of $H - P_{\mathbb{S}_+^N}(VH\,V)$ to $\mathcal{K}_2$ because

$$P_{\mathbb{S}_+^N}(VH\,V) - H = \left(P_{\mathbb{S}_+^N}(VH\,V) - VH\,V\right) + (VH\,V - H) \tag{1343}$$

The term $P_{\mathbb{S}_+^N}(VH\,V) - VH\,V$ necessarily belongs to the (dual) positive semidefinite cone by Theorem E.9.2.0.1. $V^2 = V$, hence

$$-V\left(H - P_{\mathbb{S}_+^N}(VH\,V)\right)V\succeq 0 \tag{1344}$$

by Corollary A.3.1.0.5.

Next, we require

$$\langle P_{\mathcal{K}_2}H - H\,,\; P_{\mathcal{K}_2}H\rangle = 0 \tag{1345}$$

Expanding,

$$\langle -P_{\mathbb{S}_+^N}(VH\,V)\,,\; H - P_{\mathbb{S}_+^N}(VH\,V)\rangle = 0 \tag{1346}$$

$$\langle P_{\mathbb{S}_+^N}(VH\,V)\,,\; (P_{\mathbb{S}_+^N}(VH\,V) - VH\,V) + (VH\,V - H)\rangle = 0 \tag{1347}$$

$$\langle P_{\mathbb{S}_+^N}(VH\,V)\,,\; (VH\,V - H)\rangle = 0 \tag{1348}$$

Product $VH\,V$ belongs to the geometric center subspace; (§E.7.2.0.2)

$$VH\,V\in\mathbb{S}_c^N = \{Y\in\mathbb{S}^N \mid \mathcal{N}(Y)\supseteq\mathbf{1}\} \tag{1349}$$

[6.12] $P_{\mathbb{S}_+^N}(VH\,V) = \mathbf{0}$ for $H\in\mathbb{EDM}^N$.

Diagonalize $VHV \triangleq Q\Lambda Q^{\mathrm{T}}$ (§A.5) whose nullspace is spanned by the eigenvectors corresponding to 0 eigenvalues by Theorem A.7.3.0.1. Projection of VHV on the PSD cone (§7.1) simply zeroes negative eigenvalues in diagonal matrix Λ. Then

$$\mathcal{N}(P_{\mathbb{S}_+^N}(VHV)) \supseteq \mathcal{N}(VHV)\ (\supseteq \mathcal{N}(V)) \tag{1350}$$

from which it follows:

$$P_{\mathbb{S}_+^N}(VHV) \in \mathbb{S}_c^N \tag{1351}$$

so $P_{\mathbb{S}_+^N}(VHV) \perp (VHV - H)$ because $VHV - H \in \mathbb{S}_c^{N\perp}$.

Finally, we must have $P_{\mathcal{K}_2}H - H = -P_{\mathbb{S}_+^N}(VHV) \in \mathcal{K}_2^*$. Dual cone $\mathcal{K}_2^* = -\mathcal{F}\left(\mathbb{S}_+^N \ni V\right)$ is the negative of the positive semidefinite cone's smallest face that contains auxiliary matrix V. (§6.6.1) Thus $P_{\mathbb{S}_+^N}(VHV) \in \mathcal{F}\left(\mathbb{S}_+^N \ni V\right) \Leftrightarrow \mathcal{N}(P_{\mathbb{S}_+^N}(VHV)) \supseteq \mathcal{N}(V)$ (§2.9.2.3) which was already established in (1350). ♦

From results in §E.7.2.0.2 we know: matrix product $VHV = P_{\mathbb{S}_c^N}H$ is the orthogonal projection of $H \in \mathbb{S}^N$ on the geometric center subspace $\mathbb{S}_c^N$. Thus the projection product

$$P_{\mathcal{K}_2}H = H - P_{\mathbb{S}_+^N}P_{\mathbb{S}_c^N}H \tag{1352}$$

6.8.1.1.1 Lemma. *Projection on PSD cone ∩ geometric center subspace.*

$$P_{\mathbb{S}_+^N \cap \mathbb{S}_c^N} = P_{\mathbb{S}_+^N}P_{\mathbb{S}_c^N} \tag{1353}$$

⋄

Proof. For each and every $H \in \mathbb{S}^N$, projection of $P_{\mathbb{S}_c^N}H$ on the positive semidefinite cone remains in the geometric center subspace

$$\begin{aligned} \mathbb{S}_c^N &= \{G \in \mathbb{S}^N \mid G\mathbf{1} = \mathbf{0}\} \\ &= \{G \in \mathbb{S}^N \mid \mathcal{N}(G) \supseteq \mathbf{1}\} = \{G \in \mathbb{S}^N \mid \mathcal{R}(G) \subseteq \mathcal{N}(\mathbf{1}^{\mathrm{T}})\} \qquad (1078) \\ &= \{VYV \mid Y \in \mathbb{S}^N\} \subset \mathbb{S}^N \end{aligned}$$

That is because: eigenvectors of $P_{\mathbb{S}_c^N}H$, corresponding to 0 eigenvalues, span its nullspace $\mathcal{N}(P_{\mathbb{S}_c^N}H)$. (§A.7.3.0.1) To project $P_{\mathbb{S}_c^N}H$ on the positive semidefinite cone, its negative eigenvalues are zeroed. (§7.1.2) The nullspace is thereby expanded while eigenvectors originally spanning $\mathcal{N}(P_{\mathbb{S}_c^N}H)$ remain intact. Because the geometric center subspace is invariant to projection on the PSD cone, then the rule for projection on a convex set in a subspace governs (§E.9.5, projectors do not commute) and statement (1353) follows directly. ♦

From the lemma it follows

$$\{P_{\mathbb{S}_+^N}P_{\mathbb{S}_c^N}H \mid H \in \mathbb{S}^N\} = \{P_{\mathbb{S}_+^N \cap \mathbb{S}_c^N}H \mid H \in \mathbb{S}^N\} \tag{1354}$$

Then from (2141)

$$-\left(\mathbb{S}_c^N \cap \mathbb{S}_+^N\right)^* = \{H - P_{\mathbb{S}_+^N}P_{\mathbb{S}_c^N}H \mid H \in \mathbb{S}^N\} \tag{1355}$$

$$\mathbb{EDM}^{\mathbf{2}} = \mathbb{S}_h^{\mathbf{2}} \cap \left(\mathbb{S}_c^{\mathbf{2}\perp} - \mathbb{S}_+^{\mathbf{2}}\right)$$

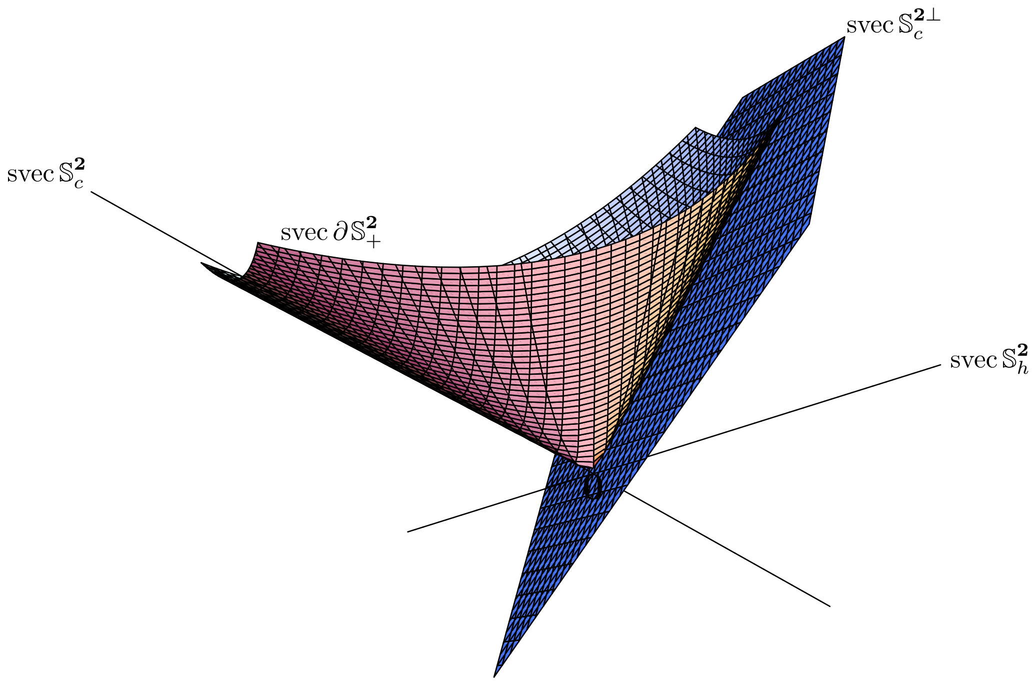


Figure 162: A plane in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$, orthogonal complement $\mathbb{S}_c^{\mathbf{2}\perp}$ (2115) (§B.2) of geometric center subspace (tiled fragment drawn) apparently supports PSD cone (rounded vertex is plot artifact). Line $\operatorname{svec}\mathbb{S}_c^{\mathbf{2}} = \operatorname{aff}\operatorname{cone}\mathcal{T}$ (1329), intersecting $\operatorname{svec}\partial\mathbb{S}_+^{\mathbf{2}}$ and drawn in Figure **156**, runs along PSD cone boundary. (*confer* Figure **143**)

$$\mathbb{EDM}^{\mathbf{2}} = \mathbb{S}_h^{\mathbf{2}} \cap \left(\mathbb{S}_c^{\mathbf{2}\perp} - \mathbb{S}_+^{\mathbf{2}}\right)$$

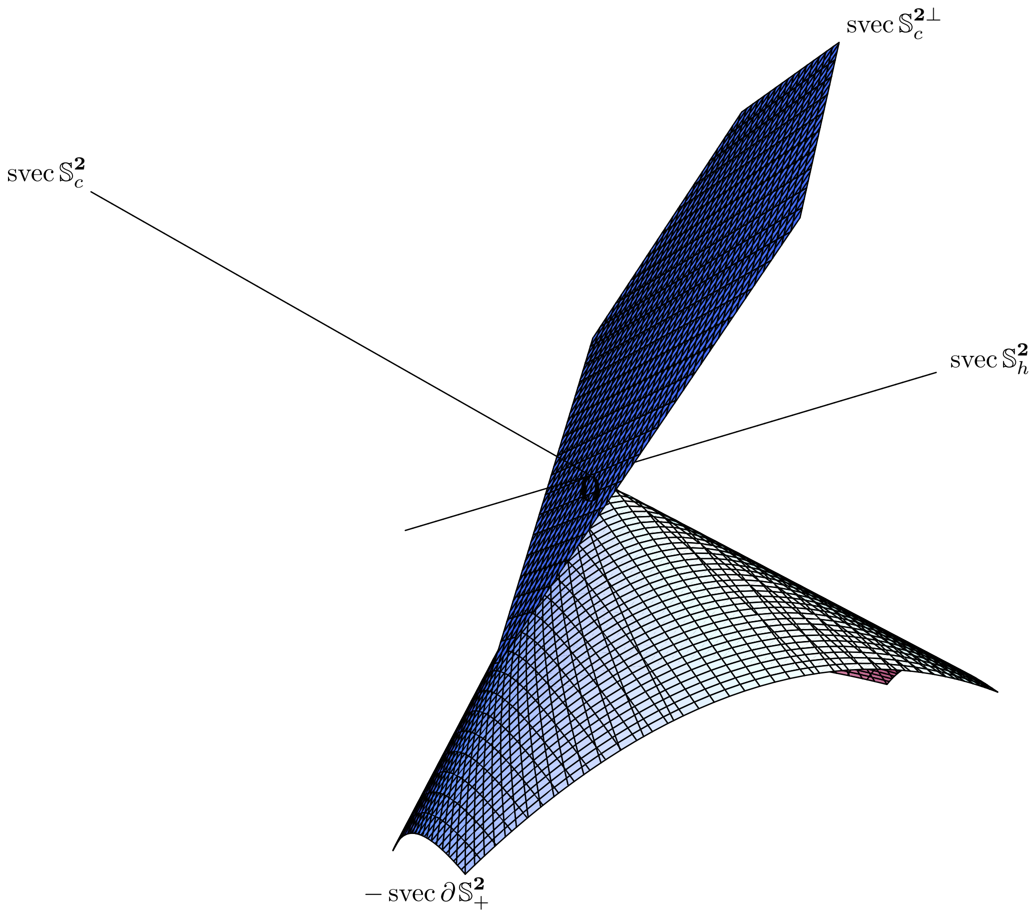


Figure 163: EDM cone construction in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$ by adding polar PSD cone to $\operatorname{svec}\mathbb{S}_c^{\mathbf{2}\perp}$. Difference $\operatorname{svec}\left(\mathbb{S}_c^{\mathbf{2}\perp} - \mathbb{S}_+^{\mathbf{2}}\right)$ is halfspace partially bounded by $\operatorname{svec}\mathbb{S}_c^{\mathbf{2}\perp}$. EDM cone is nonnegative halfline along $\operatorname{svec}\mathbb{S}_h^{\mathbf{2}}$ in this dimension.

From (313) we get closure of a vector sum

$$\mathcal{K}_2 = -\left(\mathbb{S}_c^N \cap \mathbb{S}_+^N\right)^* = \mathbb{S}_c^{N\perp} - \mathbb{S}_+^N \tag{1356}$$

therefore the equality [106]

$$\mathbb{EDM}^N = \mathcal{K}_1 \cap \mathcal{K}_2 = \mathbb{S}_h^N \cap \left(\mathbb{S}_c^{N\perp} - \mathbb{S}_+^N\right) \tag{1357}$$

whose veracity is intuitively evident, in hindsight, [93, p.109] from the most fundamental EDM definition (976).[6.13] A realization of this construction in low dimension is illustrated in Figure **162** and Figure **163**.

The dual EDM cone follows directly from (1357) by standard properties of cones (§2.13.1.1):

$$\mathbb{EDM}^{N^*} = \overline{\mathcal{K}_1^* + \mathcal{K}_2^*} = \mathbb{S}_h^{N\perp} - \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1358}$$

which bears strong resemblance to (1337).

6.8.1.2 nonnegative orthant contains $\mathbb{EDM}^N$

That $\mathbb{EDM}^N$ is a proper subset of the nonnegative orthant is not obvious from (1357). We wish to verify

$$\mathbb{EDM}^N = \mathbb{S}_h^N \cap \left(\mathbb{S}_c^{N\perp} - \mathbb{S}_+^N\right) \subset \mathbb{R}_+^{N\times N} \tag{1359}$$

While there are many ways to prove this, it is sufficient to show that all entries of the extreme directions of $\mathbb{EDM}^N$ must be nonnegative; *id est*, for any particular nonzero vector $z = [z_i\ ,\ i=1\ldots N] \in \mathcal{N}(\mathbf{1}^{\mathrm{T}})$ (§6.4.3.2),

$$\delta(zz^{\mathrm{T}})\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(zz^{\mathrm{T}})^{\mathrm{T}} - 2zz^{\mathrm{T}} \geq \mathbf{0} \tag{1360}$$

where the inequality denotes entrywise comparison. The inequality holds because the $i\,,j^{\text{th}}$ entry of an extreme direction is squared: $(z_i - z_j)^2$.

We observe that the dyad $2zz^{\mathrm{T}} \in \mathbb{S}_+^N$ belongs to the positive semidefinite cone, the doublet

$$\delta(zz^{\mathrm{T}})\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(zz^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{S}_c^{N\perp} \tag{1361}$$

to the orthogonal complement (2115) of the geometric center subspace, while their difference is a member of the symmetric hollow subspace $\mathbb{S}_h^N$. ⧫

Here is an algebraic method to prove nonnegativity: Suppose we are given $A \in \mathbb{S}_c^{N\perp}$ and $B = [B_{ij}] \in \mathbb{S}_+^N$ and $A - B \in \mathbb{S}_h^N$. Then we have, for some vector u, $A = u\mathbf{1}^{\mathrm{T}} + \mathbf{1}u^{\mathrm{T}} = [A_{ij}] = [u_i + u_j]$ and $\delta(B) = \delta(A) = 2u$. Positive semidefiniteness of B requires nonnegativity $A - B \geq \mathbf{0}$ because

$$(e_i - e_j)^{\mathrm{T}}B(e_i - e_j) = (B_{ii} - B_{ij}) - (B_{ji} - B_{jj}) = 2(u_i + u_j) - 2B_{ij} \geq 0 \tag{1362}$$

⧫

[6.13] Formula (1357) is not a matrix criterion for membership to the EDM cone, it is not an EDM definition, and it is not an equivalence between EDM operators or an isomorphism. Rather, it is a recipe for constructing the EDM cone whole from large Euclidean bodies: the positive semidefinite cone, orthogonal complement of the geometric center subspace, and symmetric hollow subspace.

6.8.1.3 Dual Euclidean distance matrix criterion

Conditions necessary for membership of a matrix $D^* \in \mathbb{S}^N$ to the dual EDM cone $\mathbb{EDM}^{N^*}$ may be derived from (1337): $D^* \in \mathbb{EDM}^{N^*} \Rightarrow D^* = \delta(y) - V_{\mathcal{N}} A V_{\mathcal{N}}^{\mathrm{T}}$ for some vector y and positive semidefinite matrix $A \in \mathbb{S}_+^{N-1}$. This in turn implies $\delta(D^* \mathbf{1}) = \delta(y)$. Then, for $D^* \in \mathbb{S}^N$

$$D^* \in \mathbb{EDM}^{N^*} \Leftrightarrow \delta(D^* \mathbf{1}) - D^* \succeq 0 \tag{1363}$$

where, for any symmetric matrix D^*

$$\delta(D^* \mathbf{1}) - D^* \in \mathbb{S}_c^N \tag{1364}$$

To show sufficiency of the matrix criterion in (1363), recall Gram-form EDM operator

$$\mathbf{D}(G) = \delta(G)\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(G)^{\mathrm{T}} - 2G \tag{988}$$

where Gram matrix G is positive semidefinite by definition, and recall the selfadjointness property of the main-diagonal linear operator δ (§A.1):

$$\langle D\,,\,D^*\rangle = \left\langle \delta(G)\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(G)^{\mathrm{T}} - 2G\,,\,D^*\right\rangle = \langle G\,,\,\delta(D^*\mathbf{1}) - D^*\rangle\, 2 \tag{1006}$$

Assuming $\langle G\,,\,\delta(D^*\mathbf{1}) - D^*\rangle \geq 0$ (1574), then we have known membership relation (§2.13.2.0.1)

$$D^* \in \mathbb{EDM}^{N^*} \Leftrightarrow \langle D\,,\,D^*\rangle \geq 0 \quad \forall\, D \in \mathbb{EDM}^N \tag{1365}$$

♦

Elegance of this matrix criterion (1363) for membership to the dual EDM cone derives from lack of any other assumptions except that D^* be symmetric:[6.14] Linear Gram-form EDM operator $\mathbf{D}(Y)$ (988) has adjoint, for $Y \in \mathbb{S}^N$

$$\mathbf{D}^{\mathrm{T}}(Y) \triangleq (\delta(Y\mathbf{1}) - Y)\, 2 \tag{1366}$$

Then from (1365) and (989) we have: [93, p.111]

$$\begin{aligned}
\mathbb{EDM}^{N^*} &= \{D^* \in \mathbb{S}^N \mid \langle D\,,\,D^*\rangle \geq 0 \quad \forall\, D \in \mathbb{EDM}^N\}\\
&= \{D^* \in \mathbb{S}^N \mid \langle \mathbf{D}(G)\,,\,D^*\rangle \geq 0 \quad \forall\, G \in \mathbb{S}_+^N\}\\
&= \{D^* \in \mathbb{S}^N \mid \left\langle G\,,\,\mathbf{D}^{\mathrm{T}}(D^*)\right\rangle \geq 0\; \forall\, G \in \mathbb{S}_+^N\}\\
&= \{D^* \in \mathbb{S}^N \mid \delta(D^*\mathbf{1}) - D^* \succeq 0\}
\end{aligned} \tag{1367}$$

the dual EDM cone expressed in terms of the adjoint operator. A dual EDM cone determined this way is illustrated in Figure **165**.

6.8.1.3.1 Exercise. *Dual EDM spectral cone.*
Find a spectral cone as in §5.11.2 corresponding to $\mathbb{EDM}^{N^*}$. ▼

[6.14] Recall: Schoenberg criterion (995) for membership to the EDM cone requires membership to the symmetric hollow subspace.

$$D^\circ = \delta(D^\circ\mathbf{1}) + (D^\circ - \delta(D^\circ\mathbf{1})) \in \mathbb{S}_h^{N\perp} \oplus \mathbb{S}_c^N \cap \mathbb{S}_+^N = \mathbb{EDM}^{N^\circ}$$

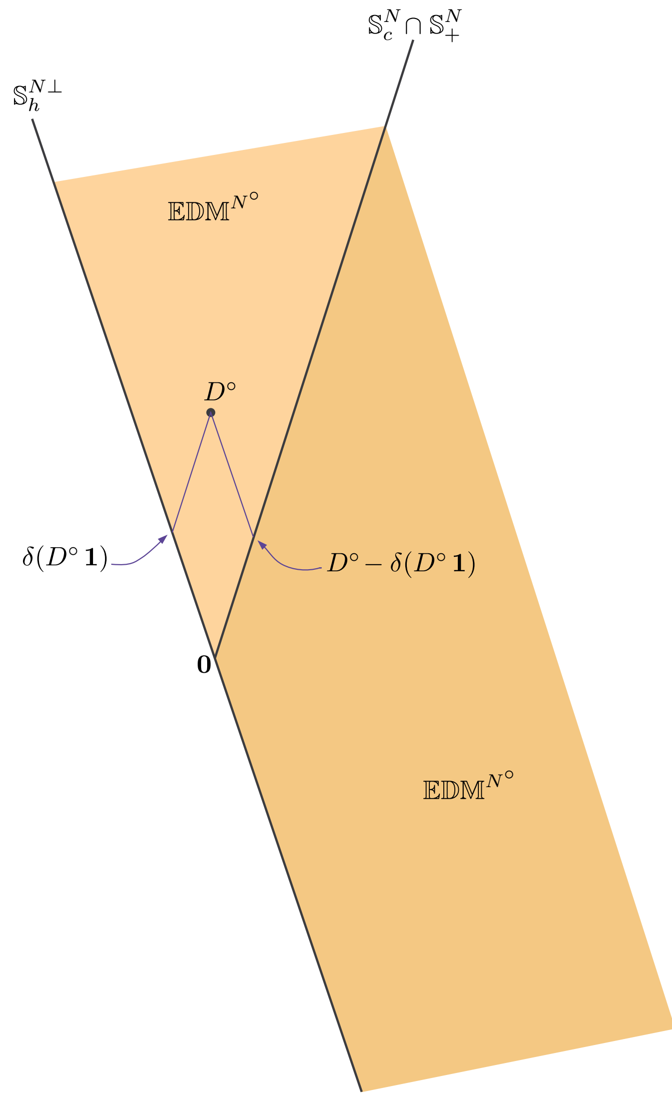


Figure 164: Hand-drawn abstraction of polar EDM cone $\mathbb{EDM}^{N^\circ}$ (drawn truncated). Any member D° of polar EDM cone can be decomposed into two linearly independent nonorthogonal components: $\delta(D^\circ\mathbf{1})$ and $D^\circ - \delta(D^\circ\mathbf{1})$.

6.8.1.4 Nonorthogonal components of dual EDM

Now we tie construct (1358) for the dual EDM cone together with the matrix criterion (1363) for dual EDM cone membership. For any $D^* \in \mathbb{S}^N$ it is obvious:

$$\delta(D^*\mathbf{1}) \in \mathbb{S}_h^{N\perp} \tag{1368}$$

any diagonal matrix belongs to the subspace of diagonal matrices (67). We know when $D^* \in \mathbb{EDM}^{N^*}$

$$\delta(D^*\mathbf{1}) - D^* \in \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1369}$$

this adjoint expression (1366) belongs to that face (1327) of the positive semidefinite cone $\mathbb{S}_+^N$ in the geometric center subspace. Any nonzero dual EDM

$$D^* = \delta(D^*\mathbf{1}) - \big(\delta(D^*\mathbf{1}) - D^*\big) \in \mathbb{S}_h^{N\perp} \ominus \mathbb{S}_c^N \cap \mathbb{S}_+^N = \mathbb{EDM}^{N^*} \tag{1370}$$

can therefore be expressed as the difference of two linearly independent $\big($when vectorized$\big)$ nonorthogonal components (Figure **143**, Figure **164**).

6.8.1.5 Affine dimension complementarity

From §6.8.1.3 we have, for some $A \in \mathbb{S}_+^{N-1}$ $\big($*confer* (1369)$\big)$

$$\delta(D^*\mathbf{1}) - D^* = V_{\mathcal{N}} A V_{\mathcal{N}}^{\mathrm{T}} \in \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1371}$$

if and only if D^* belongs to the dual EDM cone. Call $\operatorname{rank}(V_{\mathcal{N}} A V_{\mathcal{N}}^{\mathrm{T}})$ *dual affine dimension*. Empirically, we find a complementary relationship in affine dimension between the projection of some arbitrary symmetric matrix H on the polar EDM cone, $\mathbb{EDM}^{N^\circ} = -\mathbb{EDM}^{N^*}$, and its projection on the EDM cone; *id est*, the optimal solution of[6.15]

$$\begin{array}{cl} \underset{D^\circ \in\, \mathbb{S}^N}{\text{minimize}} & \|D^\circ - H\|_{\mathrm{F}} \\ \text{subject to} & D^\circ - \delta(D^\circ\mathbf{1}) \succeq 0 \end{array} \tag{1372}$$

has dual affine dimension complementary to affine dimension corresponding to the optimal solution of

$$\begin{array}{cl} \underset{D \in\, \mathbb{S}_h^N}{\text{minimize}} & \|D - H\|_{\mathrm{F}} \\ \text{subject to} & -V_{\mathcal{N}}^{\mathrm{T}} D V_{\mathcal{N}} \succeq 0 \end{array} \tag{1373}$$

Precisely,

$$\operatorname{rank}(D^{\circ\star} - \delta\big(D^{\circ\star}\mathbf{1}\big)) + \operatorname{rank}(V_{\mathcal{N}}^{\mathrm{T}} D^{\star} V_{\mathcal{N}}) = N-1 \tag{1374}$$

[6.15] This polar projection can be solved quickly (without semidefinite programming) via Lemma 6.8.1.1.1; rewriting,

$$\begin{array}{cl} \underset{D^\circ \in\, \mathbb{S}^N}{\text{minimize}} & \|\big(D^\circ - \delta(D^\circ\mathbf{1})\big) - \big(H - \delta(D^\circ\mathbf{1})\big)\|_{\mathrm{F}} \\ \text{subject to} & D^\circ - \delta(D^\circ\mathbf{1}) \succeq 0 \end{array}$$

which is the projection of affinely transformed optimal solution $H - \delta(D^{\circ\star}\mathbf{1})$ on $\mathbb{S}_c^N \cap \mathbb{S}_+^N$;

$$D^{\circ\star} - \delta(D^{\circ\star}\mathbf{1}) = P_{\mathbb{S}_+^N} P_{\mathbb{S}_c^N}\big(H - \delta(D^{\circ\star}\mathbf{1})\big)$$

Foreknowledge of an optimal solution $D^{\circ\star}$ as argument to projection suggests recursion.

and $\operatorname{rank}(D^{\circ\star} - \delta(D^{\circ\star}\mathbf{1})) \leq N-1$ because vector $\mathbf{1}$ is always in the nullspace of rank's argument. This is similar to the known result for projection on the selfdual positive semidefinite cone and its polar:

$$\operatorname{rank} P_{-\mathbb{S}_+^N} H \;+\; \operatorname{rank} P_{\mathbb{S}_+^N} H = N \tag{1375}$$

When low affine dimension is a desirable result of projection on the EDM cone, projection on the polar EDM cone should be performed instead. Convex polar problem (1372) can be solved for $D^{\circ\star}$ by transforming to an equivalent Schur-form semidefinite program (§3.5.2). Interior-point methods for numerically solving semidefinite programs tend to produce high-rank solutions. (§4.1.2) Then $D^\star = H - D^{\circ\star} \in \mathbb{EDM}^N$ by Corollary E.9.2.2.1, and $D^\star$ will tend to have low affine dimension. This approach breaks when attempting projection on a cone subset discriminated by affine dimension or rank, because then we have no complementarity relation like (1374) or (1375) (§7.1.4.1).

6.8.1.6 EDM cone is not selfdual

In §5.6.1.1, via Gram-form EDM operator

$$\mathbf{D}(G) = \delta(G)\mathbf{1}^{\mathrm{T}} + \mathbf{1}\delta(G)^{\mathrm{T}} - 2G \;\in \mathbb{EDM}^N \quad \Leftarrow \quad G \succeq 0 \tag{988}$$

we established clear connection between the EDM cone and that face (1327) of positive semidefinite cone $\mathbb{S}_+^N$ in the geometric center subspace:

$$\mathbb{EDM}^N = \mathbf{D}(\mathbb{S}_c^N \cap \mathbb{S}_+^N) \tag{1095}$$

$$\mathbf{V}(\mathbb{EDM}^N) = \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1096}$$

where

$$\mathbf{V}(D) = -VDV\tfrac{1}{2} \tag{1084}$$

In §5.6.1 we established

$$\mathbb{S}_c^N \cap \mathbb{S}_+^N = V_{\mathcal{N}}\, \mathbb{S}_+^{N-1} V_{\mathcal{N}}^{\mathrm{T}} \tag{1082}$$

Then from (1363), (1371), and (1337) we can deduce

$$\delta(\mathbb{EDM}^{N^*}\mathbf{1}) - \mathbb{EDM}^{N^*} \;=\; V_{\mathcal{N}}\, \mathbb{S}_+^{N-1} V_{\mathcal{N}}^{\mathrm{T}} \;=\; \mathbb{S}_c^N \cap \mathbb{S}_+^N \tag{1376}$$

which, by (1095) and (1096), means the EDM cone can be related to the dual EDM cone by an equality:

$$\mathbb{EDM}^N = \mathbf{D}\Big(\delta(\mathbb{EDM}^{N^*}\mathbf{1}) - \mathbb{EDM}^{N^*}\Big) \tag{1377}$$

$$\mathbf{V}(\mathbb{EDM}^N) = \delta(\mathbb{EDM}^{N^*}\mathbf{1}) - \mathbb{EDM}^{N^*} \tag{1378}$$

This means projection $-\mathbf{V}(\mathbb{EDM}^N)$ of the EDM cone on the geometric center subspace $\mathbb{S}_c^N$ (§E.7.2.0.2) is a linear transformation of the dual EDM cone: $\mathbb{EDM}^{N^*} - \delta(\mathbb{EDM}^{N^*}\mathbf{1})$. Secondarily, it means the EDM cone is not selfdual in $\mathbb{S}^N$.

6.8.1.7 Schoenberg criterion is discretized membership relation

We show the Schoenberg criterion

$$\left.\begin{array}{r}-V_{\mathcal N}^{\mathrm T} D V_{\mathcal N} \in \mathbb S_+^{N-1}\\ D \in \mathbb S_h^N\end{array}\right\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{995}$$

to be a discretized membership relation (§2.13.4) between a closed convex cone $\mathcal K$ and its dual $\mathcal K^*$ like

$$\langle y\,,\,x\rangle \geq 0 \;\text{ for all }\; y \in \mathcal G(\mathcal K^*) \;\Leftrightarrow\; x \in \mathcal K \tag{365}$$

where $\mathcal G(\mathcal K^*)$ is any set of generators whose conic hull constructs closed convex dual cone $\mathcal K^*$:

The Schoenberg criterion is the same as

$$\left.\begin{array}{r}\langle zz^{\mathrm T}, -D\rangle \geq 0 \quad \forall\, zz^{\mathrm T} \mid \mathbf{1}\mathbf{1}^{\mathrm T} zz^{\mathrm T} = \mathbf 0\\ D \in \mathbb S_h^N\end{array}\right\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{1321}$$

which, by (1322), is the same as

$$\left.\begin{array}{r}\langle zz^{\mathrm T}, -D\rangle \geq 0 \quad \forall\, zz^{\mathrm T} \in \left\{V_{\mathcal N} vv^{\mathrm T} V_{\mathcal N}^{\mathrm T} \mid v \in \mathbb R^{N-1}\right\}\\ D \in \mathbb S_h^N\end{array}\right\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{1379}$$

where the $zz^{\mathrm T}$ constitute a set of generators $\mathcal G$ for the positive semidefinite cone's smallest face $\mathcal F\left(\mathbb S_+^N \ni V\right)$ (§6.6.1) that contains auxiliary matrix V. From the aggregate in (1337) we get the ordinary membership relation, assuming only $D \in \mathbb S^N$ [215, p.58]

$$\langle D^*,\, D\rangle \geq 0 \quad \forall\, D^* \in \mathbb{EDM}^{N^*} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{1380}$$

$$\langle D^*,\, D\rangle \geq 0 \quad \forall\, D^* \in \{\delta(u) \mid u \in \mathbb R^N\} - \operatorname{cone}\left\{V_{\mathcal N} vv^{\mathrm T} V_{\mathcal N}^{\mathrm T} \mid v \in \mathbb R^{N-1}\right\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N$$

Discretization (365) yields:

$$\langle D^*,\, D\rangle \geq 0 \quad \forall\, D^* \in \{e_i e_i^{\mathrm T}\,,\; -e_j e_j^{\mathrm T}\,,\; -V_{\mathcal N} vv^{\mathrm T} V_{\mathcal N}^{\mathrm T} \mid i\,,j = 1\ldots N\,,\;\; v \in \mathbb R^{N-1}\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{1381}$$

Because $\left\langle \{\delta(u) \mid u \in \mathbb R^N\}\,,\, D\right\rangle \geq 0 \Leftrightarrow D \in \mathbb S_h^N$, we can restrict observation to the symmetric hollow subspace without loss of generality. Then for $D \in \mathbb S_h^N$

$$\langle D^*,\, D\rangle \geq 0 \quad \forall\, D^* \in \left\{-V_{\mathcal N} vv^{\mathrm T} V_{\mathcal N}^{\mathrm T} \mid v \in \mathbb R^{N-1}\right\} \;\Leftrightarrow\; D \in \mathbb{EDM}^N \tag{1382}$$

this discretized membership relation becomes (1379); identical to the Schoenberg criterion.

Hitherto a correspondence between the EDM cone and a face of a PSD cone, the Schoenberg criterion is now accurately interpreted as a discretized membership relation between the EDM cone and its ordinary dual.

6.8.2 Ambient $\mathbb{S}_h^N$

When instead we consider the ambient space of symmetric hollow matrices (1338), then still we find the EDM cone is not selfdual for $N>2$. The simplest way to prove this is as follows:

> Given a set of generators $\mathcal{G}=\{\Gamma\}$ (1298) for the pointed closed convex EDM cone, the *discretized membership theorem* in §2.13.4.2.1 asserts that members of the dual EDM cone in the ambient space of symmetric hollow matrices can be discerned via discretized membership relation:
>
> $$\begin{aligned}\mathbb{EDM}^{N^*}\cap\,\mathbb{S}_h^N \triangleq&\ \{D^*\in\mathbb{S}_h^N \mid \langle\Gamma\,,\,D^*\rangle\geq 0\quad\forall\,\Gamma\in\mathcal{G}(\mathbb{EDM}^N)\}\\ =&\ \{D^*\in\mathbb{S}_h^N \mid \langle\delta(zz^{\mathrm T})\mathbf{1}^{\mathrm T}+\mathbf{1}\delta(zz^{\mathrm T})^{\mathrm T}-2zz^{\mathrm T}\,,\ D^*\rangle\geq 0\ \ \forall\,z\in\mathcal{N}(\mathbf{1}^{\mathrm T})\}\\ =&\ \{D^*\in\mathbb{S}_h^N \mid \langle\mathbf{1}\delta(zz^{\mathrm T})^{\mathrm T}-zz^{\mathrm T}\,,\ D^*\rangle\geq 0\ \ \forall\,z\in\mathcal{N}(\mathbf{1}^{\mathrm T})\}\end{aligned}\tag{1383}$$
>
> By comparison
>
> $$\mathbb{EDM}^N = \{D\in\mathbb{S}_h^N \mid \langle -zz^{\mathrm T}\,,\ D\rangle\geq 0\ \ \forall\,z\in\mathcal{N}(\mathbf{1}^{\mathrm T})\}\tag{1384}$$
>
> the term $\delta(zz^{\mathrm T})^{\mathrm T}D^*\mathbf{1}$ foils any hope of selfdualness in ambient $\mathbb{S}_h^N$. ♦

To find the dual EDM cone in ambient $\mathbb{S}_h^N$ per §2.13.9.4 we prune the aggregate in (1337) describing the ordinary dual EDM cone, removing any member having nonzero main diagonal:

$$\begin{aligned}\mathbb{EDM}^{N^*}\cap\,\mathbb{S}_h^N =&\ \operatorname{cone}\Big\{\delta^2(V_{\mathcal N}\upsilon\upsilon^{\mathrm T}V_{\mathcal N}^{\mathrm T})-V_{\mathcal N}\upsilon\upsilon^{\mathrm T}V_{\mathcal N}^{\mathrm T}\mid \upsilon\in\mathbb{R}^{N-1}\Big\}\\ =&\ \{\delta^2(V_{\mathcal N}\Psi V_{\mathcal N}^{\mathrm T})-V_{\mathcal N}\Psi V_{\mathcal N}^{\mathrm T}\mid \Psi\in\mathbb{S}_+^{N-1}\}\end{aligned}\tag{1385}$$

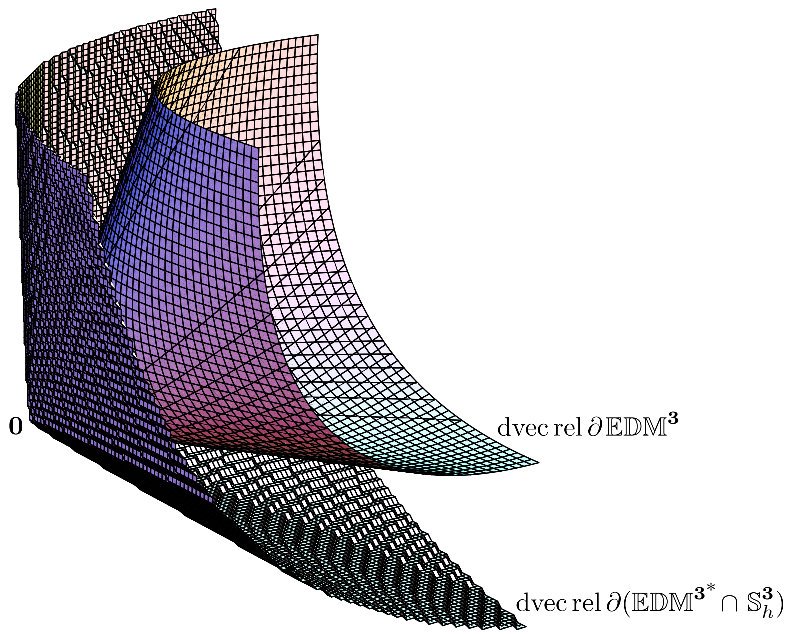
When $N=1$, the EDM cone and its dual in ambient $\mathbb{S}_h$ each comprise the origin in isomorphic $\mathbb{R}^0$; thus, selfdual in this dimension. (*confer* (104))

When $N=2$, the EDM cone is the nonnegative real line in isomorphic $\mathbb{R}$. (Figure **156**) $\mathbb{EDM}^{\mathbf{2}^*}$ in $\mathbb{S}_h^{\mathbf{2}}$ is identical, thus selfdual in this dimension. This result is in agreement with (1383), verified directly: for all $\kappa\in\mathbb{R}$, $z=\kappa\begin{bmatrix}1\\-1\end{bmatrix}$ and $\delta(zz^{\mathrm T})=\kappa^2\begin{bmatrix}1\\1\end{bmatrix}\ \Rightarrow\ d_{12}^*\geq 0$.

The first case adverse to selfdualness $N=3$ may be deduced from Figure **152**; the EDM cone is a circular cone in isomorphic $\mathbb{R}^{\mathbf{3}}$ corresponding to no rotation of Lorentz cone (178) (the selfdual circular cone). Figure **165** illustrates the EDM cone and its dual in ambient $\mathbb{S}_h^{\mathbf{3}}$; no longer selfdual.

6.8.2.0.1 Exercise. *Positive semidefinite cone from EDM cone.*
What, if any, is the inversion of semidefinite and distance cone equality (1357)? That is to say, can $\mathbb{S}_+$ be expressed only in terms of $\mathbb{EDM}$, $\mathbb{S}_h$, and $\mathbb{S}_c$? ▼

6.8.2.0.2 Exercise. *Rank complementarity for EDM cone.*
Prove (1374). ▼



$$D^* \in \mathbb{EDM}^{N^*} \Leftrightarrow \delta(D^*\mathbf{1}) - D^* \succeq 0 \qquad (1363)$$

Figure 165: Ordinary dual EDM cone projected on $\mathbb{S}_h^{\mathbf{3}}$ shrouds $\mathbb{EDM}^{\mathbf{3}}$; drawn tiled in isometrically isomorphic $\mathbb{R}^{\mathbf{3}}$. **(**It so happens: intersection $\mathbb{EDM}^{N^*} \cap \mathbb{S}_h^N$ (§2.13.9.3) is identical to projection of dual EDM cone on $\mathbb{S}_h^N$.**)**

6.9 Theorem of the alternative

In §2.13.2.1.1 we showed how alternative systems of generalized inequality can be derived from closed convex cones and their duals. This section is, therefore, a fitting postscript to the discussion of the dual EDM cone.

6.9.0.0.3 Theorem. *EDM alternative.* [179, §1]
Given $D \in \mathbb{S}_h^N$

$$\begin{array}{c} D \in \mathbb{EDM}^N \\ \text{or in the alternative} \\ \exists\ z \ \text{ such that } \left\{ \begin{array}{r} \mathbf{1}^{\mathrm{T}} z = 1 \\ Dz = \mathbf{0} \end{array} \right. \end{array} \tag{1386}$$

In words, either $\mathcal{N}(D)$ intersects hyperplane $\{z \mid \mathbf{1}^{\mathrm{T}} z = 1\}$ or D is an EDM; the alternatives are incompatible. ⋄

When D is an EDM [275, §2]

$$\mathcal{N}(D) \subset \mathcal{N}(\mathbf{1}^{\mathrm{T}}) = \{z \mid \mathbf{1}^{\mathrm{T}} z = 0\} \tag{1387}$$

Because [179, §2] (§E.0.1)

$$\begin{array}{r} DD^{\dagger}\mathbf{1} = \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} D^{\dagger} D = \mathbf{1}^{\mathrm{T}} \end{array} \tag{1388}$$

then

$$\mathcal{R}(\mathbf{1}) \subset \mathcal{R}(D) \tag{1389}$$

6.10 Postscript

We provided an equality (1357) relating the convex cone of Euclidean distance matrices to the convex cone of positive semidefinite matrices. Projection on a positive semidefinite cone, constrained by an upper bound on rank, is easy and well known; [140] simply, a matter of truncating a list of eigenvalues. Projection on a positive semidefinite cone with such a rank constraint is, in fact, a convex optimization problem. (§7.1.4)

In the past, it was difficult to project on the EDM cone under a constraint on rank or affine dimension. A surrogate method was to invoke the Schoenberg criterion (995) and then project on a positive semidefinite cone under a rank constraint bounding affine dimension from above. But a solution acquired that way is necessarily suboptimal.

In §7.3.3 we present a method for projecting directly on the EDM cone under a constraint on rank or affine dimension.